Bicycles and Tricycles: An Elementary Treatise on Their Design and Construction

by: Archibald Sharp

Published by:
MIT Press
28 Carleton St.
Cambridge, MA 02142 USA

Paper copies are $ 9.76.

Available from:
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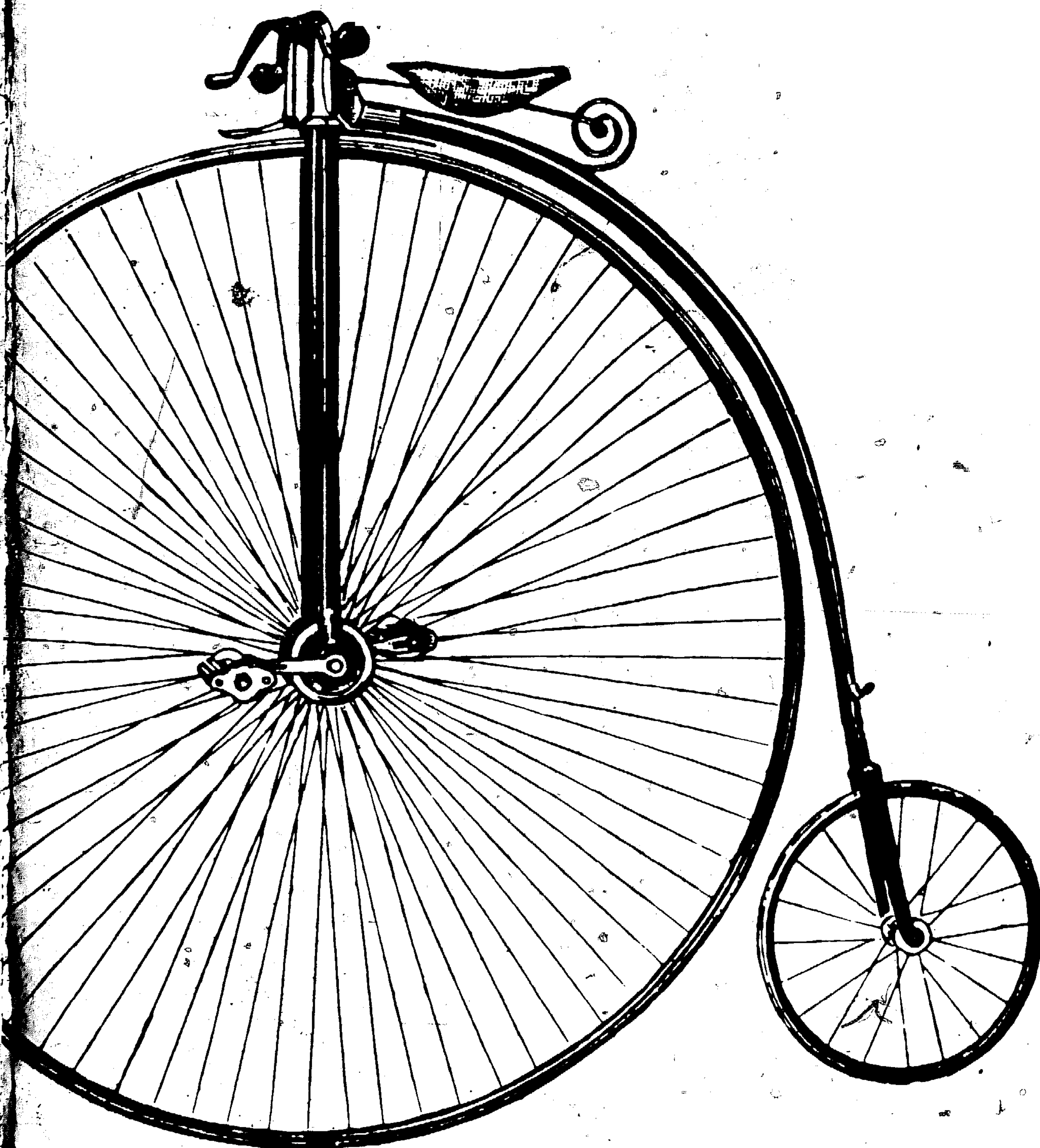

$8.95
SPORTS

Bicycles and Tricycles:
An Elementary Treatise
on Their Design and
Construction

by Archibald Sharp
Foreword to the reprint edition
by David Gordon Wilson

Published in 1896, *Bicycles and Tricycles* was the first serious, scientifically based study of the bicycle. It begins with a general exposition of mechanical principles: dynamic, static, and straining forces. It then covers successive experiments at bicycle and tricycle design, including several "mechanical monstrosities."

With the aid of elegant, sometimes humorous drawings, the book examines various designs for their relative stability, steering advantages, gearing and resistance properties. The final selection discusses the design of individual components in detail, including the frame (from the point of view of stress analysis); wheels; bearings; chains and chain gearing; toothed-wheel gearing; the lever-and-crank gear; tires; pedals, cranks and bottom brackets; springs and saddles; and brakes.

A definitive work in its own time, *Bicycles and Tricycles* is a collector's item for history-lovers as well as bicycle-enthusiasts—a treat for tinkerers and all those interested in the history of invention.

"This is a book to enjoy. Who knows, if the energy crunch gets worse it could become a best seller."—
Applied Mechanics Reviews

The MIT Press
Massachusetts Institute of
Technology
Cambridge, Massachusetts
02142

SHABP

BICYCLES & TRICYCLES

AN ELEMENTARY TREATISE ON THEIR
DESIGN AND CONSTRUCTION

WITH EXAMPLES AND TABLES

BY

ARCHIBALD SHARP, B.Sc.

WHITWORTH SCHOLAR
ASSOCIATE MEMBER OF THE INSTITUTION OF CIVIL ENGINEERS
MITGLIED DES VEREINS DEUTSCHER INGENIEURE
INSTRUCTOR IN ENGINEERING DESIGN AT THE CENTRAL TECHNICAL COLLEGE
SOUTH KENSINGTON

WITH NUMEROUS ILLUSTRATIONS

THE MIT PRESS
CAMBRIDGE, MASSACHUSETTS, AND LONDON, ENGLAND

First MIT Press paperback edition, 1979

This book was reproduced from an original in the Massachusetts Institute of Technology Libraries.

This book was printed and bound by Braun-Brumfield Company in the United States of America.

Library of Congress Cataloging in Publication Data
Sharp, Archibald, 1862–1934.
Bicycles and tricycles.

Reprint of the 1896 ed. published by Longmans, Green, London, New York.
Includes index.
1. Bicycles and tricycles. I. Title.
TL410.S5 1977 629.22'72 77–4928
ISBN 0–262–19156–3
ISBN 0–262–19156–3 (hard)
ISBN 0–262–69066–7 (paper)

FOREWORD TO REPRINT EDITION

DAVID GORDON WILSON

Archibald Sharp's definitive work on bicycles and tricycles marked, and helped to bring about, the end of an exciting period in mechanical engineering. The invention of the pedaled bicycle bore fruit after steam power had been applied successfully to road, rail, and water. But the excitement engendered by the bicycle was in some ways more widespread and intense than was inspired by the triumphs of steam. For on the one hand it emancipated large numbers of ordinary people, and particularly women, from lifetimes spent mainly within walking distances of their homes. And on the other hand it gave scope to tinkerers, blacksmiths, and engineers, allowing their imaginations full reign in conceiving new designs of machines and of components.

The result was that in the two decades following Starley's 1876 invention of the tangent-spoked tension wheel, whose light weight and high strength opened up broad areas of design freedom, an extraordinary variety of bicycles was produced. Many of these showed "utter ignorance of mechanical science," as Sharp states in his preface. One of his aims in writing this book was to make it more difficult for manufacturers to produce such "mechanical monstrosities."

And he seems to have been supremely successful. His was not the first book on bicycle design. He acknowledges his indebtedness in his preface to R. P. Scott's *Cycling Art, Energy, and Locomotion* and C. Bourlet's *Traité des Bicycles et Bicyclettes.* But Sharp's work was more thorough, complete, and authoritative — and was almost the last book as well as the last word on bicycle design. The designs he scorns here virtually disappeared after the publication of his book (it is true that many were already becoming extinct because of their own all-too-obvious inadequacies).

while the simple diamond-frame chain-driven "safety" bicycle, which he praised, ruled supreme and almost unchallenged as a utility and recreation vehicle during the following five decades of bicycle popularity all over the world.

The mass-produced, low-priced automobile running on inexpensive fuel relegated bicycles to the role of children's playthings, first in the United States in the twenties and thirties and later in Europe and other places in the fifties. But the very success of the automobile led to the end of the cheap-fuel era and to concern over the pollution and congestion resulting from over-use of automobiles in cities. A second great bicycle boom started in the United States in the late sixties and appears to be spreading to other countries. And with it has come a new outpouring of creative energy on the part of skilled engineers and of less skilled enthusiasts. Many have already wasted much effort repeating some of the mistakes Sharp pointed out in 1896. The reprinting of his book will have practical as well as historical and educational value.

His book can also be read for sheer enjoyment. He was no dry academic. It is true that at the time his book was published he was instructor in mechanical design in a prominent London college and that thereafter he was often irreverently and inaccurately referred to as "Professor" Sharp. But he spent much of his professional life as an independent consultant, inventing and developing improved machines and components for himself and for others. His salty and down-to-earth nature is very evident in the acrimonious letters he wrote to the correspondence columns of bicycling journals,* in which he attacked poor mechanical-engineering science wherever he saw it, regardless of how important, or how self-important were the perpetrators. He was almost always right.

The same forthrightness, and concern for the accuracy of engineering fundamentals, shines through every page of this book. The MIT Press will earn our gratitude for bringing back to us the complete Archibald Sharp: educator, engineer, entrepreneur, and concerned human being.

*Frank Rowland Whitt, senior author of *Bicycling Science,* sent me copies of some of Archibald Sharp's correspondence "wars" with contemporary bicycle manufacturers.

PREFACE

A BICYCLE or a tricycle is a more or less complex machine, and for a thorough appreciation of the stresses and strains to which it is subjected in ordinary use, and for its efficient design, an extensive knowledge of the mechanical sciences is necessary. Though an extensive literature on nearly all other types of machines exists, there is, strange to say, very little on the subject of cycle design; periodical cycling literature being almost entirely confined to racing and personal matters. In the present work an attempt is made to give a rational account of the stresses and strains to which the various parts of a cycle are subjected; only a knowledge of the most elementary portions of algebra, geometry, and trigonometry being assumed, while graphical methods of demonstration are used as far as possible. It is hoped that the work will be of use to cycle riders who take an intelligent interest in their machines, and also to those engaged in their manufacture.

The present type of rear-driving bicycle is the outcome of about ten years' practical experience. The old 'Ordinary,' with its large front driving-wheel, straight fork, and curved backbone, was a model of simplicity of construction,

but with the introduction of a smaller driving-wheel, driven by gearing from the pedals, and the consequent greater complexity of the frame, there was more scope for variation of form of the machine. Accordingly, till a few years ago, a great variety of bicycles were on the market, many of them utterly wanting in scientific design. Out of these, the present-day rear-driving bicycle, with diamond-frame, extended wheel-base, and long socket steering-head—the fittest—has survived. A better technical education on the part of bicycle manufacturers and their customers might have saved them a great amount of trouble and expense. Two or three years ago, when there seemed a chance of the dwarf front-driving bicycle coming into popular favour, the same variety in design of frame was to be seen; and even now with tandem bicycles there are many frames on the market which evince on the part of their designers utter ignorance of mechanical science. If the present work is the means of influencing makers, or purchasers, to such an extent as to make the manufacture and sale of such mechanical monstrosities in the future more difficult than it has been in the past, the author will regard his labours as having been entirely successful.

The work is divided into three parts. Part I. is on Mechanics and the Strength of Materials, the illustrations and examples being taken with special reference to bicycles and tricycles; Part II. treats of the cycle as a complete machine; and Part III. treats in detail of the design of its various portions.

The descriptive portions are not so complete as might be wished; however, the Cyclist Year Books,' published

early in each year, enable anyone interested in this part of the subject to be well informed as to the latest novelties and improvements.

The author would like to express his indebtedness to the following works:

The 'Cyclist Year Books';

'Bicycles and Tricycles of the Year,' by H. H. Griffin, a valuable series historically, which extends from 1878 to 1889;

'Cycling Art, Energy, and Locomotion,' by R. P. Scott;

'Traité des Bicycles et Bicyclettes,' par C. Bourlet;

The 'Cyclist' weekly newspaper;

and to the various cycle manufacturers mentioned in the text, who have, without exception, always afforded information and assistance when asked. He has also to thank Messrs. Ackermann and Farmer for assistance in preparing drawings, and Messrs. Ackermann and Hummel for reading the proofs.

In a work like the present, containing many numerical examples, it is improbable that the first issue will be entirely free from error; corrections, arithmetical and otherwise, will therefore be gladly received by the author.

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PRINCIPLES OF MECHANICS

PART I

PRINCIPLES OF MECHANICS

CHAPTER I

FUNDAMENTAL CONCEPTIONS OF MECHANICS

1. **Division of the Subject.**—*Geometry* is the science which treats of relations in space. *Kinematics* treats of space and time, and may be called the geometry of motion. *Dynamics* is the science which deals with force, and is usually divided into two parts—statics, dealing with the forces acting on bodies which are at rest; kinetics, dealing with forces acting on bodies in motion. *Mechanics* includes kinematics, statics, kinetics, and the application of these sciences to actual structures and machines.

2. **Space.**—The fundamental ideas of time and space form part of the foundation of the science of mechanics, and their accurate measurement is of great importance. The British unit of length is the *imperial yard*, defined by Act of Parliament to be the length between two marks on a certain metal bar kept in the office of the Exchequer, when the whole bar is at a temperature of 60° Fahrenheit. Several authorised copies of this standard of length are deposited in various places. The original standard is only disturbed at very distant intervals, the authorised copies serving for actual comparison for purposes of trade and commerce. The yard is divided into three *feet*, and the foot again into twelve *inches*. Feet and inches are the working units in most general use by engineers. The inch is further subdivided by engineers, by a process of repeated division by two, so that $\frac{1}{2}''$, $\frac{1}{4}''$, $\frac{1}{8}''$, $\frac{1}{16}''$, &c., are the fractions generally used by them. A more convenient

subdivision is the decimal system into $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c.; this is the subdivision generally used for scientific purposes.

The unit of length generally used in dynamics is the *foot*.

Metric System.—The metric system of measurement in general use on the Continent is founded on the *metre*, originally defined as the $\frac{1}{10,000,000}$ part of a quadrant of the earth from the pole to the equator. This length was estimated, and a standard constructed and kept in France. The metre is subdivided into ten parts called decimetres, a decimetre into ten centimetres, and a centimetre into ten millimetres. For great lengths a kilometre, equal to a thousand metres, is the unit employed.

1 metre	=	39·371 inches = 3·2809 feet.
1 kilometre	=	0·62138 miles.
1 inch	=	25·3995 millimetres.
1 mile	=	1·60931 kilometres.

3. **Time.**—The measurement of time is more difficult theoretically than that of space. Two different rods may be placed alongside each other, and a comparison made as to their lengths, but two different portions of time cannot be compared in this way. 'Time passed cannot be recalled.'

The measurement of time is effected by taking a series of events which occur at certain intervals. If the time between any two consecutive events leaves the same impression as to duration on the mind as that between any other two consecutive events, we may consider, tentatively at least, that the two times are equal. The standard of time is the *sidereal day*, which is the time the earth takes to make one complete revolution about its own axis, and which is determined by observing the time from the apparent motion of a fixed star across the meridian of any place to the same apparent motion on the following day. The intervals of time so measured are as nearly equal as our means of measurement can determine.

The *solar day* is the interval of time between two consecutive apparent movements of the sun across the meridian of any place. This interval of time varies slightly from day to day, so that for purposes of everyday life an average is taken, called the *mean solar day*. The mean solar day is about four minutes longer than

the sidereal day, owing to the nature of the earth's motion round the sun.

The mean solar day is subdivided into twenty-four *hours*, one hour into sixty *minutes*, and one minute into sixty *seconds*. The second is the unit of time generally used in dynamics.

4. **Matter.**—Another of our fundamental ideas is that relating to the existence of matter. The question of the measurement of quantity of matter is inextricably mixed up with the measurement of force. The *mass*, or quantity of matter, in one body is said to be greater or less than that in another body, according as the force required to produce the same effect is greater or less. The mass of a body is practically estimated by its weight, which is, strictly speaking, the force with which the earth attracts it. This force varies slightly from place to place on the earth's surface at sea level, and again as the body is moved above the sea level. Thus, the mass and the weight of a body are two totally different things; and many of the difficulties encountered by the student of mechanics are due to want of proper appreciation of this. The difficulty arises from the fact that the *pound* is the unit of matter, and that the *weight* of this quantity of matter, *i.e.* the force by which the earth attracts it, is used often as a unit of force. A certain quantity of lead will have a certain weight, as shown by a spring-balance, in London at high level water-mark, and quite a different weight if taken twenty thousand feet above sea level, although the mass is the same in both places.

The British unit of mass is the imperial *pound*, defined by Act of Parliament to be the quantity of matter equal to that of a certain piece of platinum kept in the office of the Exchequer.

The unit of mass in the metrical system of measurement is the *gramme*, originally defined to be equal to the mass of a cubic centimetre of distilled water of maximum density. This is, however, defined practically, like the British unit, as that of a certain piece of platinum kept in Paris.

CHAPTER II

SPEED, RATE OF CHANGE OF SPEED, VELOCITY, ACCELERATION, FORCE, MOMENTUM

5. **Speed.**—A body in relation to its surroundings may either be at rest or in motion. *Linear speed* is the rate at which a body moves along its path.

Speed may be either *uniform* or *variable.* With uniform speed the body passes over equal spaces in equal times; with variable speed the spaces passed over in equal times are unequal. The motion may be either in a straight or curved path, but in both cases we may still speak of the *speed* of a point as the rate at which it moves along its path.

6. **Uniform Speed** is measured by the space passed over in the unit of time. The unit of speed is one foot per second. Let s be the space moved over by the body moving with uniform speed in the time t, then if v be the speed, we have by the above definition.

$$v = \frac{s}{t}. \qquad (1)$$

Example.—If a bicycle move through a space of one mile in four minutes we have, reducing to feet and seconds,

$$v = \frac{3 \times 1760}{4 \times 60} = 22 \text{ feet per second.}$$

It will be seen that the unit of speed is a compound one, involving two of the fundamental units, space and time.

In the above example, the same speed is obtained whatever be the time over which we make the observations of the space described. For example, in one minute the bicycle will move

through a distance of a quarter of a mile, that is 440 yards, or 3×440 feet. Using formula (1) we get

$$v = \frac{1320}{60} = 22 \text{ feet per second,}$$

the same result as before.

Now, consider the space described by the bicycle in a small fraction of a second, say $\frac{1}{10}$th, if the speed is uniform, this will be 2·2 ft. Using formula (1) again, we have

$$v = \frac{2 \cdot 2}{\frac{1}{10}} = 22 \text{ feet per second.}$$

Proceeding to a still smaller fraction of a second, say $\frac{1}{1000}$th, if our means of observation were sufficiently refined, the distance passed over in the time would evidently be found to be the $\frac{22}{1000}$th part of a foot, *i.e.* = ·022 feet. Again using formula (1) we have

$$v = \frac{\cdot 022}{\frac{1}{1000}} = 22 \text{ feet per second.}$$

Uniform Motion in a Circle.—Another familiar example of uniform motion is that of a point moving in a circular path; a point on the rim of a bicycle wheel has, relative to the frame of the bicycle, such a motion, uniform when the speed of the bicycle is uniform. The linear speed, relative to the frame, of a point on the extreme outside of the tyre will be the same as the linear speed of the bicycle along and relative to the road, while that of any point nearer the centre of the wheel will be less.

7. **Angular Speed.**—When a wheel is rotating about its axis, the linear speed of any point on it depends on its distance from the centre, is greatest when the point is on the circumference of the wheel, and is zero for a point on the axis. The number of complete turns the wheel, as a whole, makes in a second gives a convenient means of estimating the rotation. Let O (fig. 1) be the centre of a wheel, and A a point on its circumference; OA may thus represent the position of a spoke of the wheel at a certain instant. At the end of one second, suppose the spoke which was initially in the position OA_1 to occupy the position OA_2; if the motion of rotation of the wheel is uniform, the linear

speed of the point A on the rim is measured by the arc $A_1 A_2$, while the angular speed of the wheel is measured by the angle $A_1 O A_2$. Generally, the angular speed of a body rotating uniformly is the angle turned through in unit of time.

The angular speed may be expressed in various ways. For example, the number of degrees in the angle $A_1 O A_2$ swept out per second may be expressed; this method, however, is little used practically. The method of expressing angular velocity most in use by engineers, is to give the number of revolutions per minute, n.

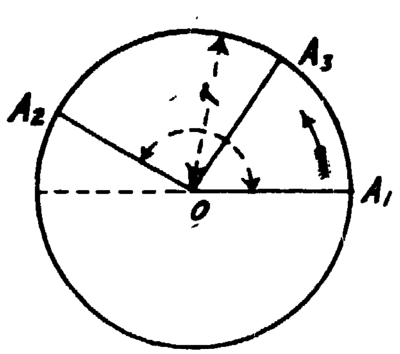


FIG. 1.

One revolution $= 360°$; revolutions per minute can be converted into degrees per second by multiplying by 360 and dividing by 60, that is, by multiplying by 6.

For scientific purposes another method is used. Mathematicians find that the most convenient unit angle to adopt is not obtained by dividing a right angle into an arbitrary number of parts; they define the unit angle as that which subtends a circular arc of length equal to the radius. Thus, in figure 1, if the arc $A_1 A_3$ be measured off equal to the radius $O A_1$, the angle $A_1 O A_3$ will be the *unit angle*. This is called a *radian*.

The ratio of the length of the circumference of a circle to its diameter is usually denoted in works on mathematics and mechanics by the Greek letter π (pronounced like the English word 'pie'), and is 3.14159 . . . . This number is 'incommensurable,' which means that it cannot be expressed exactly in our ordinary system of numeration. It may, however, be expressed with as great a degree of accuracy as is desired; a very rough value often used for caculations is $3\frac{1}{7}$. It is easily seen that there are π radians in an angle of half a revolution, and therefore the angle of one revolution, that is, four right angles, is 2π radians. Therefore, 1 radian $= \frac{360}{2\pi} = 57{\cdot}28°$.

The angular speed ω of a rotating body is expressed in radians turned through per second, and

$$\omega = \frac{2\pi n}{60} \quad \ldots\ldots\ldots \quad (2)$$

8. **Relation between Linear and Angular Speeds.**—The connection between the angular speed of a rotating body and the linear speed of any point in it may now be easily expressed. Let O (fig. 1) be the centre of the rotating body, and A a point on it, distant r from the centre, which moves in unit of time from A_1 to A_2, the number of units in the linear speed of A is equal to the number of units in the length of the arc $A_1 A_2$, similarly the angular speed of the rotating body is numerically equal to the angle $A_1 O A_2$ in radians. But this by definition must be equal to the arc $A_1 A_2$ divided by the radius $O A_1$, hence if ω (omega) be the angular speed of a rotating body, v the linear speed of any point on it distant r from the centre, we have

$$\omega = \frac{v}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The speed of a bicycle is conveniently expressed in miles per hour, and the angular speed of the driving-wheel in revolutions per minute. Let V be the speed in miles per hour, D the diameter of the driving-wheel in inches, and n the number of revolutions of the driving-wheel per minute; then feet and seconds being the units in (3),

$$\omega = \frac{2\pi n}{60}, \quad v = \frac{V \times 5280}{3600}, \quad r = \frac{D}{2 \times 12}.$$

Substituting in (3) we get

$$\frac{2\pi n}{60} = \frac{V \times 5280 \times 2 \times 12}{3600 \times D},$$

from which $$V = \frac{n D}{336 \cdot 13} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

that is, the speed of the bicycle in miles per hour is equal to the number of revolutions per minute of the driving-wheel, multiplied by the diameter of the driving-wheel in inches, and divided by 336·13.

A more convenient rule than the above for finding the speed of a bicycle can be deduced. Let N be the number of revolutions of the driving-wheel made in t seconds; then

$$N = \frac{n \times t}{60}, \text{ and } n = \frac{60 N}{t}.$$

Substituting in (4), we get

$$V = \frac{60\ N D}{336{\cdot}13\ t}.$$

Now, suppose that N be chosen equal to V; that is, t is chosen such that the number of revolutions in t seconds is equal to the number of miles travelled in one hour. Substituting above we get

$$t = \frac{D}{5{\cdot}502}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which is equivalent to the following convenient rule. Divide the diameter of the driving-wheel in inches by 5·502, the number of revolutions of the driving-wheel made in the number of seconds equal to this quotient is equal to the speed of the cycle in miles per hour.

If, in a geared-up cycle, D be taken as the diameter to which the driving-wheel is geared, N will be the number of revolutions per minute of the crank-axle, and formula (5) will still apply.

9. **Variable Speed.**—The numerical example in section 6 may help towards a clear understanding of the measurement of variable speed. When the speed of a moving body is changing from instant to instant, if we want to know the speed at a certain point, it would be quite incorrect to observe the space described by the body in, say, one hour or one minute after passing the point in question ; but the smaller the interval of time chosen, the more closely will the *average* speed during that interval approximate to the speed *at the instant* of passing the point of observation.

Now; suppose the body after passing the point to move with exactly the same speed it had at the point, and that in t seconds it moves over s feet, its speed at the point of observation would be $\frac{s}{t}$ feet per second. In a very small fraction of a second, say $\frac{1}{1000}$th, the amount of change in the speed of the body is very small, and by taking a sufficiently small period of time the average speed during the period may be considered equal to the speed at the beginning of the period, without any appreciable error. The

speed at any instant will thus be expressed by equation (1), *provided t be chosen small enough.*

Suppose a bicyclist just starting to race, and that we wish to observe his speed at a point 5 feet from the starting-point. We observe the instant he passes the point, and the distance he travels in a period of time reckoned from that instant. If in a minute he travel 2,400 feet, his *average* speed during that time $= \frac{2400}{60} = 40$ feet per second. But in a quarter-minute, reckoned from the same instant, he may only travel 420 feet, giving an average speed of $\frac{420}{15} = 28$ feet per second; while in five seconds he may only have travelled 110 feet, in one second 15 feet, in one-tenth of a second 1·05 foot, in one-hundredth part of a second one-tenth part of a foot, with average speeds during these periods of 22, 15, 10·5, and 10 feet per second. The last of these values may be taken as a very close approximation to his speed when passing the point in question.

10. **Velocity.**—If the speed of a point and the direction of its motion be known, its *velocity* is defined: thus, in the conception 'velocity,' those of 'speed' and 'direction' are involved. Velocity has been defined as 'speed directed,' or 'rate of change of position.' Again, speed may be defined as the *magnitude* of velocity.

Velocity, involving as it does the idea of direction, can therefore be represented by a straight line, the direction of which indicates the direction of the motion, and, by choosing a suitable scale, the length of the line may represent the speed, or the magnitude of the velocity. A quantity which has not only magnitude and algebraical sign, but also direction, is called a *vector* quantity. Thus, velocity is a vector quantity. A quantity which has magnitude and sign, but is independent of direction, is called a *scalar* quantity. Speed is a scalar quantity.

Velocity may be *linear* or *angular*; it may also be uniform or variable. A point on a body rotating with uniform angular speed about a fixed axis has its linear speed uniform, but since the direction of its motion is continually changing, its linear velocity is variable, its angular velocity is uniform. *Angular velocity* can

also be represented by a vector, the direction of the vector being parallel to the axis of rotation, and the length of the vector being equal to the angular speed.

11. **Rate of Change of Speed.**—If a moving body at a certain instant has a speed of 3 feet per second, and a second later a speed of 4 feet per second, two seconds later a speed of 5 feet per second, three seconds later a speed of 6 feet per second, and so on; in one second the speed increases by 1 foot per second. In other words, its rate of change of speed is 1 foot per second per second.

Rate of change of speed may be either uniform or variable. An important example of uniform rate of change of speed is that of a body falling freely under the action of gravity. If a stone be dropped from a height, its speed at the instant of dropping is zero; at the end of one second, as determined by experiment, 32·2 feet per second approximately; at the end of two seconds, 64·4 feet per second; at the end of three seconds, 96·6 feet per second—at least, these would be the speeds if the air offered no resistance to the motion. Thus the rate of change of speed of a body falling freely under the action of gravity is 32·2 feet per second per second.

If a be the rate of change of speed of a body starting from rest, at the end of t seconds its speed will be

$$v = a\,t \qquad \dots\dots\dots\dots \qquad (6)$$

Its *average* speed during the time will be $\frac{1}{2}\,a\,t$, and therefore the space it passes over in time t is

$$s = \tfrac{1}{2}\,a\,t \times t = \tfrac{1}{2}\,a\,t^2 \qquad \dots\dots \qquad (7)$$

A cyclist starting in a race affords a good example of variable rate of change of speed. At the instant of starting the speed of the machine and rider is zero; at the end of two seconds it may be five miles an hour; at the end of three seconds, nine miles an hour; at the end of four seconds, thirteen; at the end of five, seventeen; at the end of six, twenty; at the end of seven, twenty-two; at the end of eight, twenty-three; at the end of nine, twenty-three and threequarters—the increase in the speed with each second becoming smaller and smaller until, say fifteen or twenty

seconds from the start, the maximum speed is reached, the speed remains constant, and the rate of change becomes zero. In this case not only the speed, but also its rate of change, is variable. The rate of change probably increases at first, and reaches its maximum soon after the start, then diminishes, and ultimately reaches the value zero. If the speed of a body diminish, its rate of change of speed is negative. A cyclist while pulling up previous to stopping is moving with negative rate of change of speed.

The unit of rate of change of speed, like that of speed, is a compound one, into which the fundamental units of time and space enter. In expressing rate of change of speed we have used the phrase 'feet per second per second'; this deserves careful study on the part of the beginner, as a proper understanding of the ideas involved in these units is absolutely necessary for satisfactory progress in mechanics. This rate of change is often loosely spoken of in some of the earlier text books as so many 'feet per second'; this method of expression is quite wrong. For instance, considering the rate of change of speed due to gravity, we have stated above that it is 32 feet per second per second. This means that at the end of one second the speed of a freely falling body is increased by an additional speed of 32 feet per second, or 1,920 feet per minute. In one minute the speed would be increased by sixty times the above additional speed—that is, by 1,920 feet per second, or 115,200 feet per minute. This rate of change of speed may therefore be expressed either as '32 feet per second per second,' '1,920 feet per minute per second,' or '115,200 feet per minute per minute.'

The relation between the units of rate of change of speed, space, and time is expressed by the formula (7), which may be written

$$a = \frac{2s}{t^2},$$

which shows that the magnitude of the unit rate of change of speed is proportional to that of unit space, and inversely proportional to the square of that of unit time.

12. **Rate of Change of Angular Speed.**—The angular speed of a rotating body may be either constant or variable; in the

latter case the *rate of change of angular speed* is the increment in one unit of time of the angular speed. Let θ be the rate of change of angular speed, a the rate of change of linear speed of any point on the body distant r from the centre, then

$$\theta = \frac{a}{r} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

13. **Acceleration** is rate of change of velocity; it may be zero, uniform, or variable. When it is zero the velocity remains constant, and the motion takes place in a straight line.

When a point is moving with uniform speed in a circle, though its speed does not change, the direction of its motion changes, and therefore its velocity also changes. It must therefore be subjected to acceleration. An acceleration which does not change the speed of the body on which it acts must be in a direction at right angles to that of the motion, and is called *radial* acceleration. An acceleration which does not change the direction of a moving body must act in the direction of motion, and is called *tangential* acceleration. The *magnitude* of the tangential acceleration is the rate of change of speed.

14. **Force.**—The definition and measurement of force has afforded scope for endless metaphysical disquisitions. Force has been defined as 'that which produces or tends to produce motion in a body.' The unit of force is defined as 'that force which, acting for one unit of time on a body initially at rest, produces at the end of the unit of time a motion of one unit speed.' If the units of space, mass, and time be one foot, one pound, and one second respectively, the unit of force is called a *poundal.* In the centimetre-gramme-second system of units, the unit of force is called a *dyne.* The measurement of the unit of mass involves the idea of force, so that perhaps no satisfactory logical definition can be given.

The unit of force above defined is called the *absolute unit.* The magnitude of a force in absolute units is measured by the acceleration it would produce in unit of time on a body of unit mass. The force with which the earth attracts one pound of matter is equal to 32·2 poundals, since in one second it produces an acceleration of 32·2 feet per second per second. Generally,

if a force f acting on a mass m produces an acceleration a, we have

$$f = m\,a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The unit of force used for practical purposes is the *weight* of one *pound* of matter; this is called the gravitation unit of force. If f be the number of absolute, and F the number of gravitation units in a force, $f = gF$, or

$$F = \frac{f}{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The acceleration due to gravity is usually denoted by the letter g. The value of g, or, in other words, the weight of unit mass in absolute units of force, as has already been stated above, varies from place to place on the earth's surface. For Britain its value is approximately 32·2, the foot-pound-second system of units being used.

Great care must be exercised in distinguishing between one pound quantity of matter and 1 lb. weight, the former being a unit of mass, the latter an arbitrary unit of force.

15. **Momentum.**—The product of the mass of a body into its velocity is called its *quantity of motion* or *momentum*. The momentum of a body of mass one pound moving with a velocity of ten feet per second, is thus the same as that of a body of mass ten pounds moving with a velocity of one foot per second.

16. **Impulse.**—Multiply both sides of equation (9) by t, we then get

$$f\,t = m\,a\,t.$$

But if the body start from rest, $a\,t = v$, its velocity at the end of t seconds, therefore

$$f\,t = m\,v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Equation (11) asserts therefore that the momentum, $m\,v$, of a body initially at rest is equal to the product of the force acting on it and the time during which the force acts. The product $f\,t$ is called the *impulse* of the force.

Equation (11) is true, however small t, the time during which the force acts, may be. Now a momentum of 10 foot-pounds per second may be generated by the application of a force of 1 lb. acting for ten seconds, or a force of ten poundals for one second, or a force of 1000 poundals acting for $\frac{1}{100}$th part of a

second; and so on. When two moving bodies collide, or when a blow is struck by a hammer, the surfaces are in contact for a very small fraction of a second, and the mutual force between the bodies is very great. Neither the force nor the time during which it acts can be directly measured, but the momentum of the bodies before and after collision can be easily measured. Such forces of great magnitude acting for a very short space of time are called *impulsive* forces; they differ only in degree, but not in kind, from forces acting for appreciably long periods.

17. **Moments of Force, of Momentum, &c.**—Let figure 2 represent a body fastened by a pin at O, so that it is free to turn about O as a centre, but is otherwise constrained. Let it be acted on by the forces P_1 and P_2. Now, it is a matter of every day experience that the turning effect of such a force as P_1 depends not only on its magnitude, but also, in popular language, on its leverage, that is, on the length of the perpendicular from the centre of rotation to the line of action of the force.

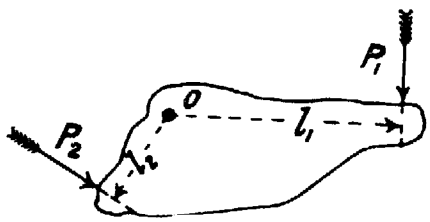


FIG. 2.

For example, in screwing up a nut, if a long spanner be used the force required to be exerted at its end is much smaller than if a short spanner be used. The product $P_1\, l_1$ of the force into this distance is called the *moment* of the force about the given centre. The force P_1 tends to turn the body in the direction of the hands of a watch, while P_2 tends to turn the body in the opposite direction. Therefore, if the moment $P_2\, l_2$ be considered positive, the moment $P_1\, l_1$ must be considered negative. The positive direction is usually taken *contra-clockwise*.

If the body be at rest under the action of the forces P_1 and P_2 their moments must be equal in magnitude but of opposite sign; that is, their algebraic sum must be zero.

The *moment of momentum* about a given point O of a body of mass m moving with velocity v is the product of its momentum $m\, v$, and the length of the perpendicular l from the given point to the direction of motion—*i.e.*, $m\, v\, l$. In the same way the moment of an impulse $f\, t$ is the product of the impulse and the length of the perpendicular from the given point to the line of action of the impulse—*i.e.*, $f\, t\, l$.

CHAPTER III

KINEMATICS; ADDITION OF VELOCITIES

18. **Graphic Representation of Velocity.**—For the complete specification of a velocity two elements—its *magnitude* and *direction*—are necessary. If a body be moving at any instant with a certain velocity, the direction of the motion may be represented by the direction of a straight line drawn on the paper, and the speed of the body by the length of the straight line. For this purpose the unit of speed is supposed to be represented by any convenient length on the paper: the number of times this length is contained in the straight line drawn will be numerically equal to the speed of the body. For example, the line $a\,b$ (fig. 3) represents a velocity of three feet per second in the direction of the arrow, while the line $c\,d$ represents a velocity of two feet per second in a direction at right angles to that of the former. The scale of velocity in this diagram has been taken 1 foot per second = ¼ inch. In the same way, any quantity which involves direction as well as magnitude can be represented by a straight line having the same direction and its length proportional to some scale to the magnitude of the quantity. Such a straight line is called a *vector*.

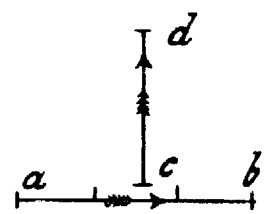


FIG. 3.

Example.—If a wheel be turning about its axis with uniform speed, the velocities of all points on its rim are numerically equal, but have all different directions. Thus, the velocities of the points A, B, and C on the rim (fig. 4) are represented by the three equal lines, $A\,a$, $B\,b$, and $C\,c$ respectively at right angles to the radii $O\,A$, $O\,B$, and $O\,C$.

19. **Addition of Velocities.**—A body may be subjected at

the same instant to two or more velocities, and its aggregate velocity may be required. For example, take a man climbing the mast of a ship. Let the ship move horizontally in the direction $a\,b$ (fig. 5), and let the length $a\,b$ indicate the space passed over by it in one second. Let $a\,c$ be the mast, and as it passes the point a let the man commence climbing. At the end of one second suppose he has climbed the distance $a\,d$. The line $a\,d$ will represent the velocity of the man climbing up the mast, the line $a\,b$ the velocity of the ship. But if $a\,c$ be the position of the mast at the beginning of the second, at the end of the second it will be in the position $b\,c^1$, and the man will be at d^1, the length $b\,d^1$ being, of course, the same as $a\,d$. The actual velocity, in

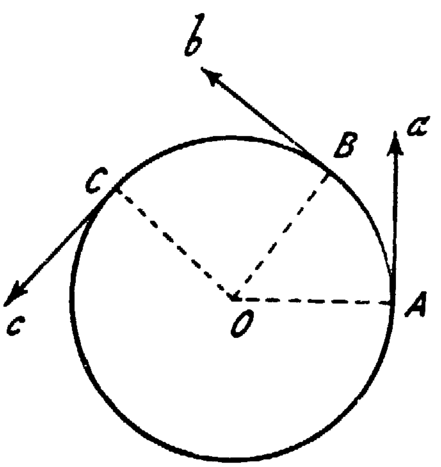


FIG. 4.

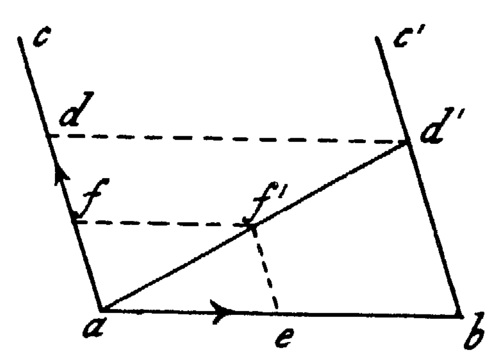


FIG. 5.

magnitude and direction, of the man is represented by the line $a\,d^1$. At the end of half a second the foot of the mast would be at e, $a\,e$ being equal to $\frac{1}{2}\,a\,b$, and the man would have ascended the mast a distance $a\,f$; the actual position of the man would be f^1, midway between a and d^1. Thus his actual motion in space will be along the line $a\,d^1$.

The two velocities $a\,b$ and $a\,d$ above are called the *component* velocities, and the velocity $a\,d^1$ the *resultant* velocity of the man.

20. **Relative Velocity.**—We have spoken above of the motion of a body, meaning thereby the motion of the body in relation to the objects in its immediate neighbourhood, but these objects themselves may be in motion in relation to some other objects. For example, a man walking from window to window of

a railway carriage in rapid motion has a motion of a certain velocity relative to the carriage. But the carriage itself is in motion relative to the earth, and the motion of the man relative to the earth is quite different from that relative to the carriage. Again, the earth itself is not at rest, but rotates on its own axis, so that the man's motion relative to a plane of fixed direction passing through the earth's axis is still more complex. But besides a motion round its own axis, the earth has a motion round the sun. The sun itself, and with it the whole solar system, has a motion of translation relative to the visible universe; in fact, there is no such thing as absolute rest in nature. Therefore, having no body at rest to which we can refer the motion of any body, we know nothing of absolute motion. The motions we deal with, therefore, are all relative, and the velocities are also relative. It will thus often be necessary, in specifying a velocity to express the body in relation to which it is measured.

21. **Parallelogram of Velocities.**—Given two component uniform velocities to which a body is subjected, the resultant velocity of the body may therefore be found as follows:—Draw a parallelogram with two adjacent sides, *o a* and *o b* (fig. 6),

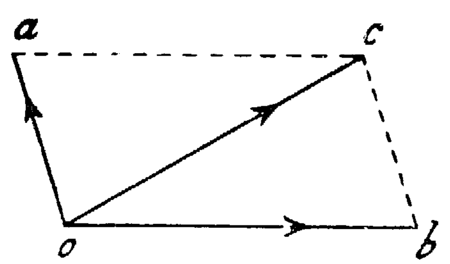


FIG. 6.

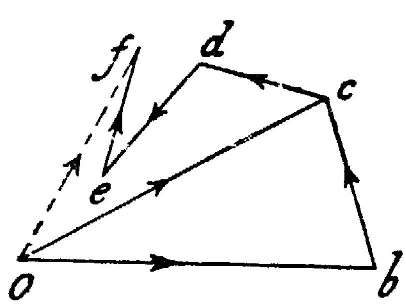


FIG. 7.

representing in magnitude and direction the component velocities The resultant velocity is represented in magnitude and direction by the diagonal *o c* of the parallelogram. This proposition is known as *the parallelogram of velocities*. Since velocity involves the two ideas of speed and direction, but not position, the resultant of two velocities may also be found by the following method:—Let *o b* (fig. 7) be one of the given velocities; from *b* draw *b c* equal and parallel to the other; *o c* will represent the resultant velocity.

Vector Addition.—The geometrical process used above is

called 'vector addition,' and is used in compounding any physical quantities that can be represented by, and are subject to the same laws as, vectors. Accelerations, forces acting at a point, rotations about intersecting axes, are treated in this way. In general, the sum of any number of vectors is obtained by placing at the final point of one the initial point of another, and so forming an unclosed irregular polygon ; the vector formed by joining the initial point of the first to the final point of the last is the required sum, the result being independent of the order in which the component vectors are taken. Thus, the sum of the vectors $o\,b$, $b\,c$, $c\,d$, $d\,e$, and $e\,f$ (fig. 7) is the vector $o\,f$.

22. **Velocity of any Point on a Rolling Wheel.**—Let a wheel roll along the ground, its centre having the velocity v. The wheel as a whole partakes of this velocity, which may be

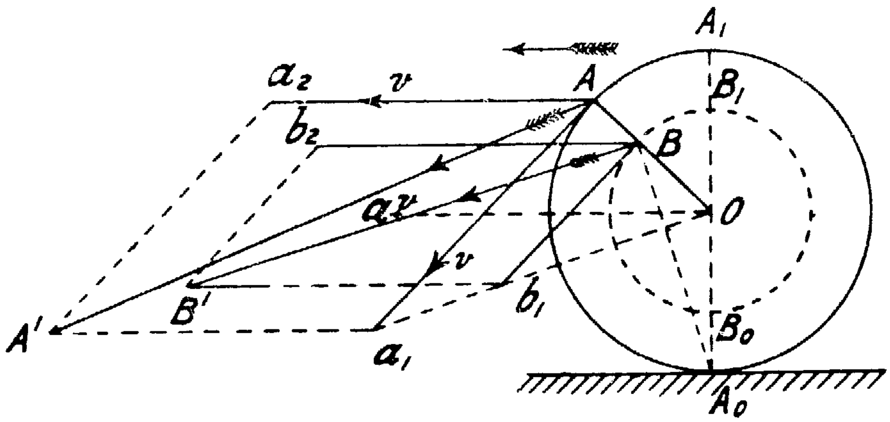


FIG. 8.

represented by the line $O\,a$ (fig. 8). The relative motion of the wheel and ground will be the same if we consider the centre of the wheel fixed and the ground to move backwards with velocity v. In this way it is seen that the linear speed of any point on the rim of the wheel relative to the frame is equal to v. We can now find the velocity, relative to the earth, of any point A on the rim of the wheel. The point A is subjected to the horizontal forward velocity $A\,a_2$ with speed v, and to the velocity with speed v, in a direction $A\,a_1$ at right angles to $O\,A$, due to the rotation of A round O. The resultant velocity is obtained by completing the parallelogram $A\,a_1\,A^1\,a_2$. The diagonal $A\,A^1$ represents the velocity of A in magnitude and direction. If the point on the rim be taken at A_0, the point of contact with the ground, it will be seen that the parallelogram in the above construction reduces to

two coincident straight lines. In this case, however, it is easily seen that the velocity of A, due to the rotation of the wheel, is backwards, and, therefore, when added to the forward velocity due to the translation of the wheel, the resultant velocity is zero. On the other hand, if the point be taken at A_1, the top of the wheel, the velocity due to rotation is in the forward direction. Thus, the velocity of the uppermost point of the wheel is $2v$—that is, twice the velocity of the centre.

In the same way the velocity of any point B on one of the spokes may be found. Join $O\ a_1$ and draw $B\ b_1$ parallel to $A\ a_1$, meeting $O\ a_1$ at b_1. The velocity of B, due to rotation, is represented by $B\ b_1$. Draw $B\ b_2$ equal and parallel to $A\ a_2$, and complete the parallelogram $B\ b_2\ B^1\ b_1$. The velocity of B is represented by $B\ B^1$. It will be found that the velocity of B is greatest when passing its topmost position B_1, and least when passing its lowest position B_0.

The above problem can be dealt with by another method. The motion of the wheel has been compounded of two motions, the linear motion of the bicycle and the motion of rotation of the wheel about its axis. But the resultant motion of the wheel—that is, its motion relative to the ground—can be more simply expressed. If the wheel rolls on the ground without slipping, its point of contact A_0 is, at the moment in question, at rest. The linear velocity of the wheel's centre O is evidently the same as that of the bicycle v, and is in a horizontal direction. The centre of the wheel, therefore, may be considered to rotate about the point A_0. But as the wheel is a rigid structure, every point on it must be rotating about the same centre. The point A_0 is called the *instantaneous centre of rotation* of the wheel. The linear velocity of any point on the wheel is, by (3) (chap. ii.), equal to $\omega\ r$, where r is the distance of the point from the centre of rotation A_0. But ω is equal to $\dfrac{v}{r_0}$, where r_0 is the radius of the wheel; therefore, the linear velocity of any point B on the wheel is equal to $\dfrac{v}{r_0}.A_0B$, and is in the direction $B\,B^1$ at right angles to $A_0\ B$.

The centre of rotation A_0 of the wheel is called an *instan-*

taneous centre of rotation, as distinguished from a *fixed* centre of rotation, since when the wheel is rolled through any distance however small, its point of contact with the ground, and therefore its centre of rotation, is changed.

23. **Resolution of Velocities** is the converse of the addition of velocities, and has for its object the finding of components in two given directions, whose resultant motion shall be equal to the given motion. If *o c* (fig. 6) be the given velocity, *o b* and *o a* the directions of the required components, the latter are found by drawing from *c* straight lines, *c b* and *c a*, parallel respectively to *o a* and *o b*, cutting them at *b* and *a* : *o b* and *o a* represent the required components.

Example.—Suppose a cyclist to ride up an incline of one in ten at the rate of ten miles an hour. To find at what rate he rises vertically, draw a horizontal line *A B* (fig. 9) ten inches long, and a vertical line *B C* one inch long; join *A C*. Along this line to any convenient scale mark off *A D*, the velocity ten miles an hour ($14\frac{2}{3}$ feet per second). Draw *D E* at right angles to *A B*, meeting *A B*, produced, if necessary, at *E*. *D E* is the required vertical velocity of the cyclist. By measurement this is found to be 1·46 feet per second (less than 1 mile per hour).

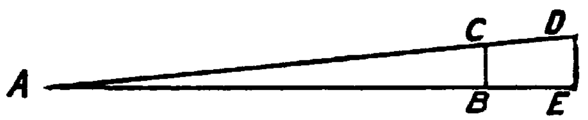


FIG. 9.

Example.—A cyclist is riding along the road with a velocity indicated in direction and magnitude by *O A* (fig. 10). The wind is blowing with velocity *O B*, and is therefore partially in the direction in which the cyclist is riding. To find the apparent direction of the wind, that is, its direction relative to the moving bicycle, join *A B* and draw *O C* equal and parallel to *A B*; *O C* will be the velocity of the wind apparent to the cyclist, which is thus apparently blowing partially against him. The velocity *O C* can be resolved into two, *O D* dead against the cyclist, and *D C* sideways, *C D* being drawn at right angles to *A O*. For, from the parallelogram of velocities it is seen that the actual velocity, *O B*, of the wind relative to the earth is compounded of its velocity

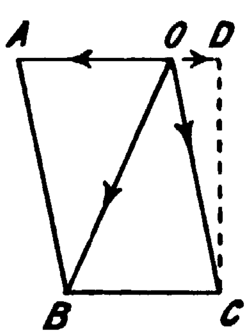


FIG. 10.

relative to the bicycle $O\,C$, and the velocity of the bicycle, $O\,A$, relative to the earth.

The above figure may explain why cyclists seldom seem to feel a back wind, while head winds seem always to be present.

24. **Addition and Resolution of Accelerations.**—An acceleration involves the idea of magnitude and direction, but not position; it may, therefore, be represented by a vector. Figs. 6 and 7 are, therefore, directly applicable to the compounding and resolving of accelerations.

25. **Hodograph**.—If a body move in any path, its velocity at any instant, both as to direction and magnitude, can be conveniently represented by a vector drawn from a fixed point; the curve formed by the ends of such vector is called the *hodograph* of the motion.

26. **Uniform Circular Motion.**—The hodograph for uniform circular motion can easily be found as follows:—When the body is at A (fig. 11), its velocity is in the direction $A\,A^1$. From a fixed point o (fig. 12) set off $o\,a$ equal and parallel to $A\,A^1$. When the body is at B its velocity is represented by $B\,B^1$, equal in length to $A\,A^1$; the corresponding line $o\,b$ on the hodograph (fig. 12) is equal and parallel to $B\,B^1$. Repeating this construction for a number of positions of the moving body, it is seen that the hodograph $a\,b\,c$ is a circle.

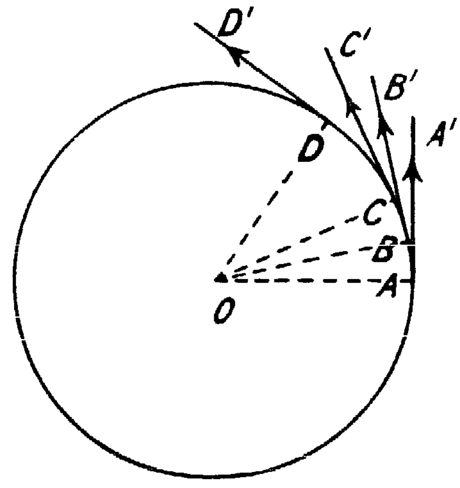


FIG. 11.

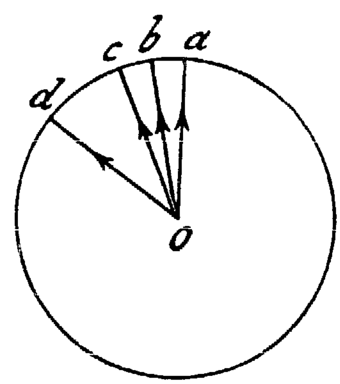


FIG. 12.

Since the direction of motion changes from instant to instant, the moving body must be subjected to an acceleration, which can be determined as follows:—When the body is at A, its velocity is represented by $o\,a$, and when at B by $o\,b$; therefore, in the interval of passing from A to B an additional velocity, represented by $a\,b$, has been impressed on it. If the point B be taken

very close to A, *i.e.* if a very short interval of time be taken, b will be very close to a, and therefore $a\,b$, the direction of the impressed velocity, will be parallel to $A\,O$, *i.e.* directed towards the centre of rotation. If the interval of time is taken sufficiently small, the additional velocity $a\,b$ is also very small, and the resultant velocity $o\,b$ does not sensibly differ in magnitude from $o\,a$; thus the only effect of the additional velocity is to change the direction of motion from $o\,a$ to $o\,b$ (fig. 12).

When at B suppose the body to undergo the same operation, at the end of it the direction of the motion will be $o\,c$. After a number of such operations the body will be at D (fig. 11), and its velocity will be represented by $o\,d$ (fig. 12). The total additional velocity imparted to it between the positions A and D has only had the effect of changing the direction, but not the magnitude of its velocity. This total additional velocity is represented by the arc $a\,d$.

Now, suppose the body to take one second to pass from A to D, then $a\,d$ represents the increase of velocity in unit of time, and is, therefore, numerically equal to the acceleration α. Let v be the linear speed of the body, and r the radius of the circle in which it moves; then the arc $A\,D$ is numerically equal to v, $o\,a$ is by definition equal to v, and since $o\,a$ and $o\,d$ are respectively parallel to the tangents at A and D, the angle $a\,o\,d$ is equal to the angle $A\,O\,D$; therefore,

$$\frac{\alpha}{v} = \frac{\text{arc } a\,d}{\text{radius } a\,o} = \frac{\text{arc } A\,D}{\text{radius } A\,O} = \frac{v}{r}.$$

$$i.e. \qquad \alpha = \frac{v^2}{r} \qquad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (1)$$

That is, in uniform circular motion, the radial acceleration is proportional to the square of the speed, and inversely proportional to the radius.

CHAPTER IV

KINEMATICS—PLANE MOTION

27. **Definitions of Plane Motion.**—If a body move in such a manner that each point of it remains always in the same plane, it is said to have *plane motion*. Plane motion can be perfectly represented on a flat sheet of paper: and, fortunately for the engineer, most moving parts of machines have only plane motion. In cycling mechanics there are more examples of motion in three dimensions. The motion of the wheels as the machine is moving in a curve and the motion of a ball in its bearing are examples of non-plane motion.

Each particle of a body having plane motion will describe a plane curve, which is called a *point-path*.

28. **General Plane Motion of a Rigid Body.**—The plane motion of a rigid body may be—

(1) *Simple translation*, without rotation. In this case any straight line drawn on the rigid body always remains in the same direction. The motion of the body will be completely determined if that one point of it is known.

(2) *Rotation about a fixed axis.*—In this case the path of any point is a circle of radius equal to the distance of the point from the axis of rotation.

(3) *Translation combined with a motion of rotation.*—We shall see later that in this case it is possible to represent the motion *at any instant* by a rotation of the body about an axis perpendicular to the plane of motion, the position of the axis, however, changing from instant to instant.

If the paths of two points of a rigid body be known, the path of any other point on the rigid body is determined. Let A, B, and C (fig. 13) be three points rigidly connected, A moving on

the curve $a\,a$, B on the curve $b\,b$. The path of the point C can evidently be found as follows:—Let A_1 be any position of the point on the curve $a\,a$; the corresponding position B_1 is found by drawing an arc with centre A_1 and radius $A\,B$, cutting the curve $b\,b$ in B_1. With centres A_1 and B_1, and radii $A\,C$ and $B\,C$ respectively, draw two arcs intersecting at C_1. C will be a point on the path described by C.

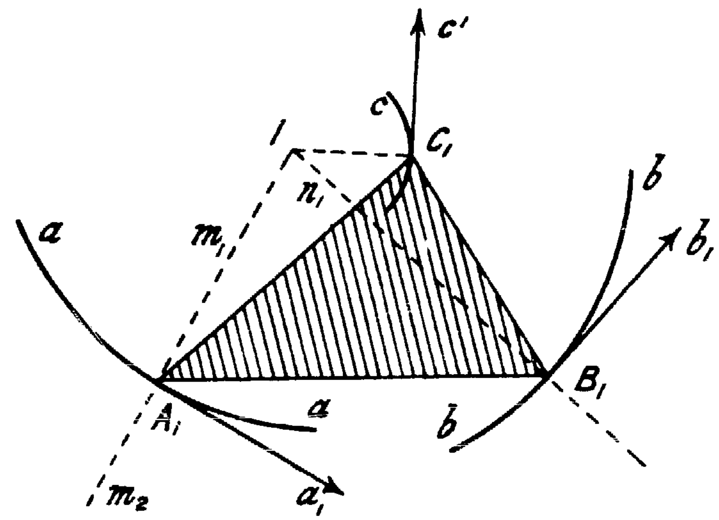


FIG. 13.

29. **Instantaneous Centre.**—Let A and B (fig. 13) be two points of a rigid body, $a\,a$ and $b\,b$ their respective point-paths. In the position shown the direction of the motion of A is a tangent at the point A_1 to the curve $a\,a$. The point A may therefore be considered to rotate about any point, m, lying on the normal at A_1 to the curve $a\,a$. For, if A be considered to rotate either about m_1 or m_2, the direction of the motion at the instant is in either case the same tangent, $A_1\,a_1$. In the same way, since the tangent $B_1\,b_1$ is also the tangent to any circle through B having its centre on the normal $B_1\,n_1$, the point B may be considered to rotate about any point in the normal at B_1 to the curve $b\,b$. If the normals $A_1\,m_1$ and $B_1\,n_1$ intersect at I, A and B may be both considered to be rotating at the instant about the centre I. No other point in the plane satisfies this condition, I is therefore called the *instantaneous centre of rotation* of the rigid body. Every point on the rigid body is at the instant rotating about the centre I, therefore the tangent at C_1 to the point-path $c\,c$ is at right angles to $C_1\,I$.

30. **Point-paths, Cycloidal Curves.**—A few point-paths described in simple mechanisms are of great importance in mechanics. We will briefly notice the most important.

Cycloid.—If a circle roll, without slipping, along a straight line, the curve described by a point on its circumference is called

a *cycloid.* Let a circle roll along the straight line XX (fig. 14). The curve described by the point C on its circumference can be readily drawn as follows: Divide the circumference of the circle into a number of equal parts (twenty-four will be convenient, as this division can be effected by the use of the 45° and 60° set squares), and number the divisions as shown. Through the centre draw a straight line parallel to XX; this will be the path of the centre of the circle. Along this line mark off a number of divisions, each equal in length to those on the circumference of the circle, and number them correspondingly. When any point,

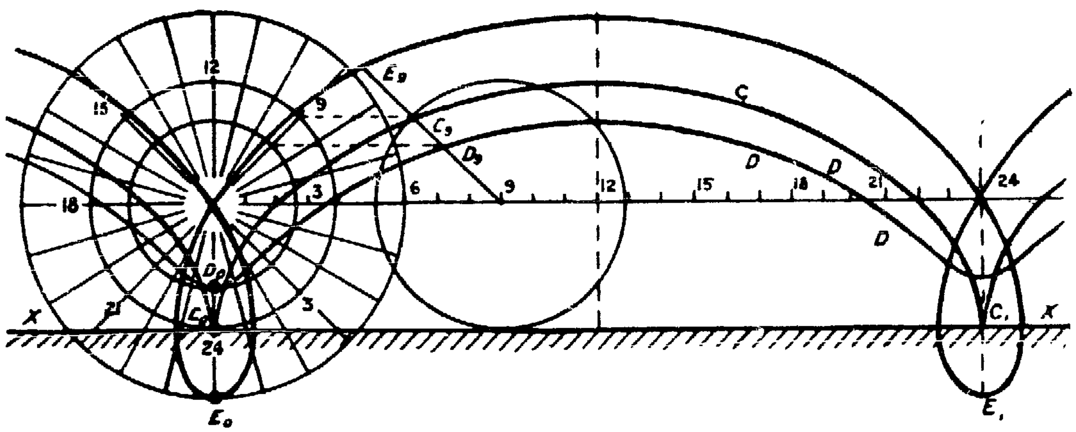


FIG. 14.

say 9, on the circumference of the circle is rolled into contact with the line XX, the centre of the circle will be on the corresponding point, 9, of the straight line. Draw the circle in this position. The corresponding position C_9 of C is evidently obtained by projecting over from the point 9 of the circumference. By repeating this process for each of the points of division, twenty four points on the cycloid will be obtained; through these a fair curve may be drawn freehand. The curve $C_0\,C\,C_1$ shows one portion of the cycloid. The point-path is a repetition, time after time, of this curve.

Prolate and Curtate Cycloid.—The path described by a point, D, inside the rolling circle is called a *prolate* cycloid. $D_0\,D\,D_1$ shows one complete portion of the curve. The method of drawing it is exhibited in figure 14, and hardly requires any further explanation.

The curve described by a point lying outside the rolling circle is called a *curtate* cycloid. $E_0\,E\,E_1$ (fig. 14) shows one complete portion.

A point on the circumference of a bicycle wheel describes a cycloid as the machine moves in a straight line. Any point on the spokes, or any point on the crank, describes a prolate cycloid.

Epicycloid and Hypocycloid.—If one circle roll on the circumference of another, the curve described by a point on the circumference of the rolling circle is called an *epicycloid* or a *hypocycloid*, according as the rolling circle lies outside or inside the fixed circle. These curves are of great importance in the theory of toothed-wheels.

In figure 15, EE is an epicycloid and HH a hypocycloid, in each of which the diameter of the rolling circle is one-third that of the fixed circle. The method of drawing these curves is similar to that of drawing the cycloid, the only difference being that the divisions along the path of the centre of the rolling circle will not be equal to those along the circumference of the rolling circle, but the divisions along the fixed and rolling circles will correspond.

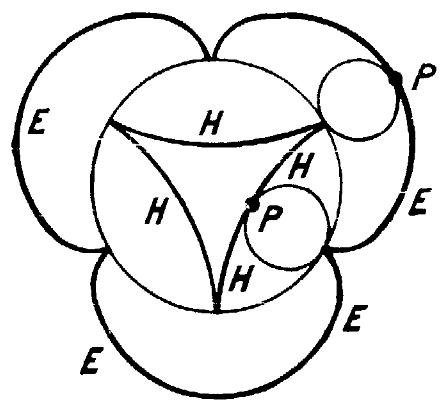


FIG. 15.

A particular case occurs when the diameter of the rolling circle is equal to the radius of the fixed circle ; the hypocycloid in this case reduces to a straight line, a diameter of the fixed circle.

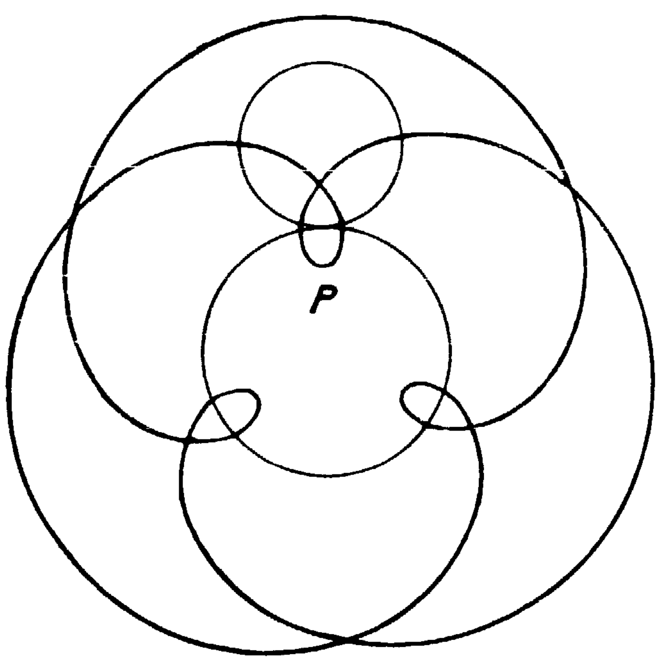


FIG. 16.

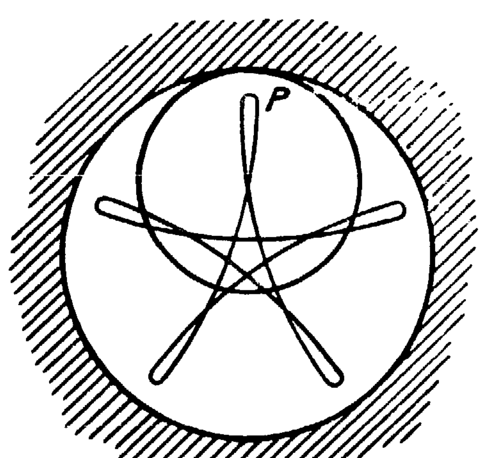


FIG. 17.

Epitrochoids and Hypotrochoids.—If the tracing point does not lie on the circumference of the rolling circle, the curve traced is called an *epitrochoid* or a *hypotrochoid*. Figures 16, 17, and 18 show some examples of epitrochoids and hypotrochoids.

Involute.—Let a string be wrapped round a circle and have a pencil attached at some point; as it is unwound from the circle

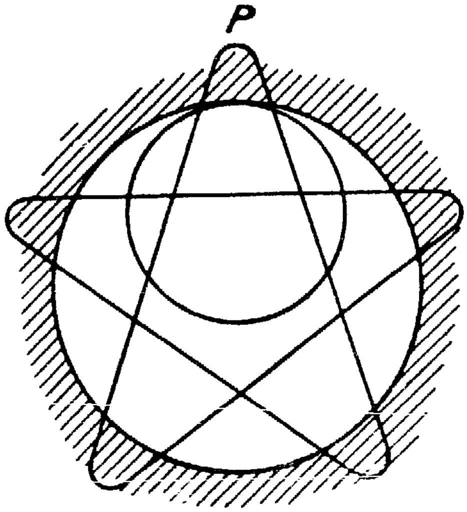


FIG. 18.

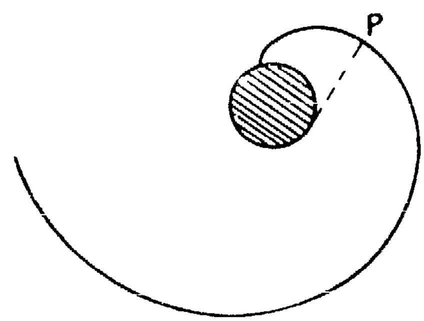


FIG. 19.

the pencil will describe a curve on the paper, called an *involute* (fig. 19). This curve is also of great importance in the theory of toothed-wheels.

The involute is a particular case of an epicycloid. If the rolling circle be of infinitely great radius its circumference will become a straight line. The curve traced out by a point P (fig. 19) of a straight line, which rolls without slipping on a circle, is an involute.

31. **Point-paths in Link Mechanisms.**—We have already shown how to find the path described by any point of a rigid body of which two point-paths are known. If the paths $a\,a$ and $b\,b$ (fig. 13) be circular arcs, the bar AB may be considered as the coupling link between two cranks. The variety of curves described by points rigidly connected to such a coupling link is very great; some of them have been of great practical use. Figure 20 shows a point-path described

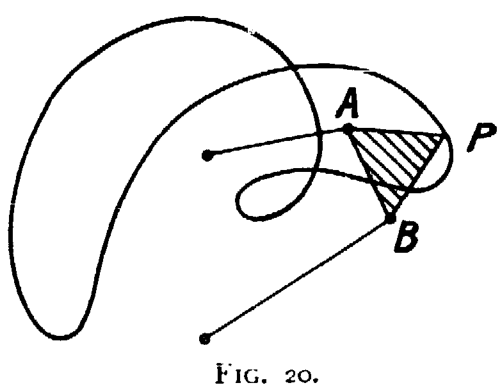


FIG. 20.

by a tracing point, P, which does not lie on the axis of the link $A B$.

In Singer's 'Xtraordinary' bicycle the motion given to the pedal was such a curve. The mechanism and the path described by the pedal are discussed in chapter xxix.

32. **Speeds in Link Mechanisms.**—If the speed of any point in a mechanism be known, it will in general be possible to determine that of any other point. In a four-link mechanism, $A B C D$ (fig. 21), in which $C D$ is the fixed link, the nature of the motion will depend on the relative length of the links. If $D A$ be the shortest, $A B + D A$ be less than $C D + B C$, and $A B - D A$ be greater than $C D - B C$, $D A$ will rotate continuously, and $C B$ oscillate. The speeds of points on the lever $C B$ are proportional to their distances from the fixed centre of rotation C; similarly for points on the lever $D A$. Now in any position of the mechanism the link $A B$ may be considered to have a rotation about the instantaneous centre I, the point of intersection of $A D$ and CB, produced if necessary; and thus the linear speed of any point of the link is proportional to its distance from I. If the point A rotates with uniform speed, the point B will oscillate in a circular arc with a variable speed. Let v_a be the uniform speed of A, and v_b the corresponding speed of B. Then, since the body $A B$ is rotating at the instant about the centre I,

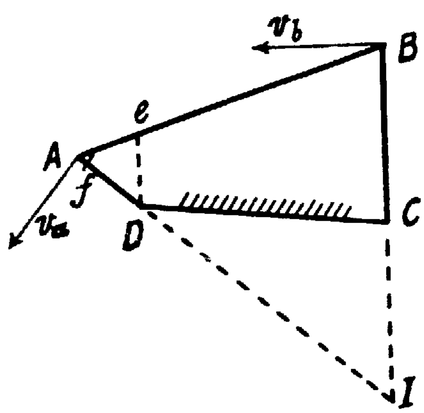


FIG. 21.

$$\frac{v_a}{v_b} = \frac{I A}{I B}.$$

Draw $D e$ parallel to $C B$, meeting $A B$, produced if necessary, at e. Then the triangles $A D e$, $A I B$ are similar, and therefore

$$\frac{I A}{I B} = \frac{D A}{D e},$$

and
$$\frac{v_b}{v_a} = \frac{D e}{D A},$$

or
$$v_b = \frac{v_a}{D A} D e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now $D A$ is constant whatever be the position of the mechanism, and therefore if v_a, the speed of A, be constant, the speed of the point B is proportional to the intercept $D\ e$.

Mark off $D f$ along $D A$ equal to $D e$. The length $D f$ is thus proportional to the speed of the point B when the crank $D A$ is in the corresponding position. If this construction be repeated for all positions of the crank $D A$, the locus of the point f will be the *polar curve* of the speed of the point B.

33. **Speed of Knee-joint when Pedalling a Crank.**—In pedalling a crank-driven cycle, the motion of the leg from the hip

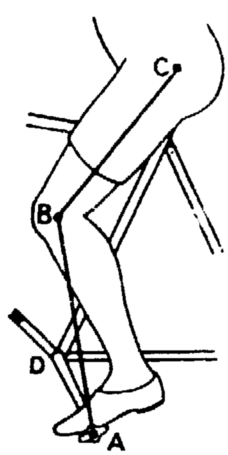


FIG. 22.

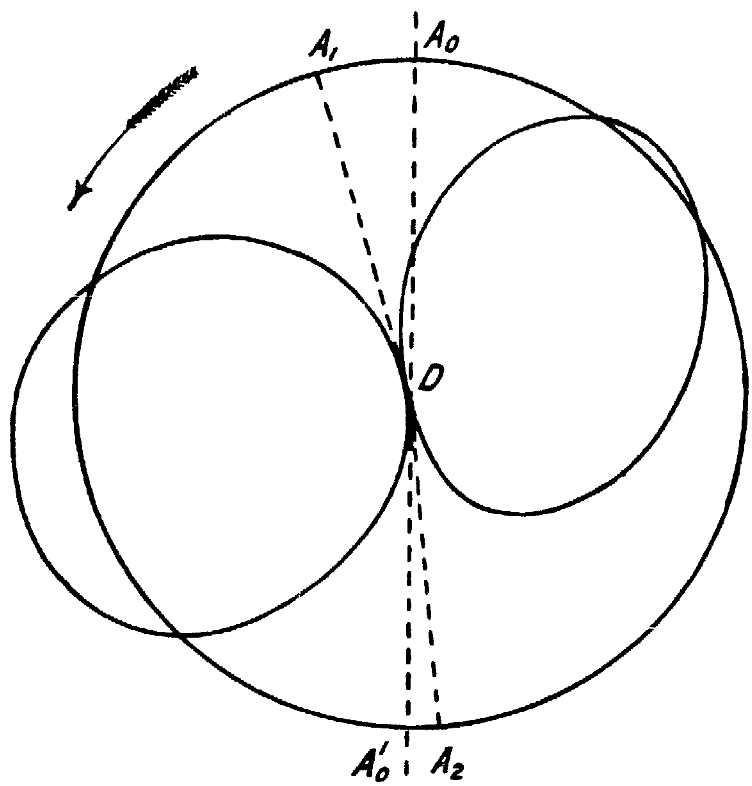


FIG. 23.

to the knee is one of oscillation about the hip-joint. If the ankle be kept quite stiff during the motion, as, unfortunately, is too often the case with beginners, the leg from the knee-joint downwards practically constitutes the coupling-link of a four-link mechanism. The pedal-pin (fig. 22) rotating with uniform speed, figure 23 shows the curve of speed of the knee-joint. It may be noticed that the maximum speed of the knee during the up-stroke is less than during the down-stroke. Also, the point B is at the upper end of its path when the pedal-pin is in the position A_1, some considerable distance after the vertical position $D\ A_0$ of the crank ; while B is in its lowest position when the pedal pin is at

A_2. The angle $A_1 D A_2$, passed through by the crank during the down-stroke of the knee is less than the angle passed through during the up-stroke; consequently, since the speed of the pedal-pin is uniform, the average speed of the knee during the down-stroke is less than during the up-stroke. If the rider can just barely reach the pedal when at its lowest point, the speed of the knee-joint is very great immediately before and after coming to rest at the lowest point of its path.

34. **Simple Harmonic Motion.**—If P be a point moving with uniform speed in a circle of radius r of which $a\ b$ is a diameter, and $P\ p$ be a perpendicular let fall on $a\ b$ (fig. 24), while P moves in the circle, the point p will move backwards and forwards along the straight line $a\ b$. The point p is then said to have *simple harmonic motion*. The motion of a point on a vibrating string, and of a particle of air in an organ-pipe when the simplest possible, is of this character. The speed of p will vary with its varying position. At any instant the velocity of the point P is in the direction $P\ m$, the tangent at P. Setting off $v = P\ m$ to scale along this line it may be resolved into two components $P\ n$ and $n\ m$ respectively parallel, and at right angles, to $a\ b$. The parallel component $P\ n$ is, of course, equal to the speed of the point p. If the scale of v be taken such that $P\ m$ is equal to r, the triangles $P\ m\ n$ and $P\ o\ p$ are equal, and therefore $P\ p$ is equal to $P\ n$. That is, in any position of the point p moving along $a\ b$ with simple harmonic motion, its speed may be represented by the ordinate $p\ P$ to the circle on $a\ b$ as diameter.

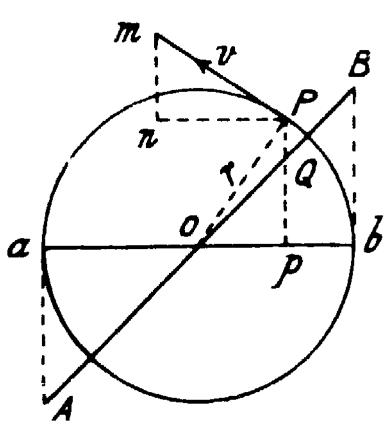


FIG. 24.

If P moves uniformly in the circle, its acceleration is constant in magnitude and equal to $\frac{v^2}{r}$, and is in the direction of the radius $P\ o$. The scale of acceleration may be chosen such that the vector $P\ o$ represents the acceleration of P, which may be decomposed into $P\ p$ and $p\ o$ respectively at right angles, and parallel to, $a\ b$. The parallel component $p\ o$ is, of course, equal to the acceleration of the point p along $a\ b$—that is, in simple harmonic

motion the acceleration is proportional to the distance of the moving point from the centre of its motion. If an ordinate $p\,Q$ be set off equal to $o\,p$, the locus of Q will be the acceleration diagram of the motion; this locus is a straight line $A\,B$ passing through o, the centre of the motion.

The motion of the knee-joint when pedalling approximates to simple harmonic motion, the approximation being closer the shorter the crank $D\,A$ (fig. 22) is in comparison with the lever $C\,B$ and connecting-link $A\,B$. If the motion were exactly simple harmonic motion, the polar curve of speed of knee-joint (fig. 23) would consist of two circles passing through D.

35. **Resultant Plane Motion.**—*Resultant of Two Translations.*—If a rigid body be subjected to two motions of translation simultaneously, the resultant motion will evidently be a motion of simple translation, which can be found by an application of the parallelogram of velocities.

Resultant of Two Rotations about Parallel Axis.—Let a body be subjected to two rotations, ω_1 and ω_2, about the axes A and B

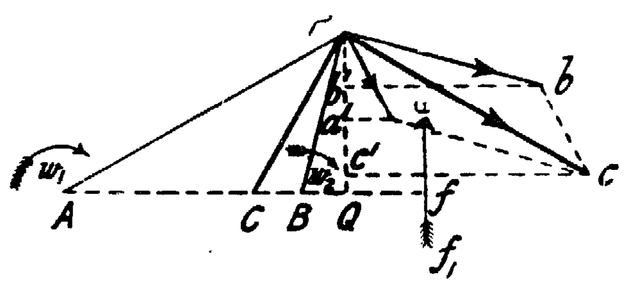


FIG. 25.

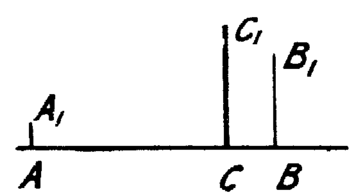


FIG. 26.

(fig. 25). If the motion be plane, the axes must be parallel, and at right angles to the plane of the motion. Let P be any point in the body. Join P to A and B, and draw $P\,a$ and $P\,b$ at right angles to $P\,A$ and $P\,B$ respectively. The resultant linear velocity of P will be the resultant of the velocity $\omega_1 \times A\,P$ in the direction $P\,a$, and of $\omega_2 \times \overline{B\,P}$ in the direction $P\,b$. If $P\,a$ and $P\,b$ be marked off respectively equal to these velocities to any convenient scale, the resultant $P\,c$ can be found by the parallelogram of velocities.

From P draw a perpendicular $P\,Q$ on the line, produced if necessary, joining the centres A and B. Draw $a\,a^1$ and $b\,b^1$ perpendicular to $P\,Q$. Then, the velocity of P due to the rotation ω_1 about A may be resolved into the velocity $a^1\,a$ parallel to, and the velocity $P\,a^1$ at right angles to, $A\,B$. Similarly, the velocity of P

due to the rotation ω_2 about B may be resolved into the two components $P\,b^1$ and $b^1\,b$. The triangles $A\,Q\,P$ and $P\,a^1\,a$ are similar; so, also, are the triangles $B\,Q\,P$ and $P\,b^1\,b$. It is, therefore, easy to show that the components of P's velocity due to ω_1, at right angles, and parallel, to $A\,B$, are respectively $(\omega_1 \times A\,Q)$ and $(\omega_1 \times Q\,P)$. Similarly, the components due to ω_2 are $(\omega_2 \times B\,Q)$ and $(\omega_2 \times \overline{Q\,P})$. Therefore, the components of P's resultant velocity at right angles, and parallel, to $A\,B$ are respectively :—

$$v_1 = (\omega_1 \times A\,Q) + (\omega_2 \times B\,Q) \quad . \quad . \quad . \quad . \quad (2)$$

and

$$v_2 = (\omega_2 + \omega_1)\,P\,Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let C be a point on the straight line $A\,B$, dividing it in the inverse ratio of the angular speeds ω_1 and ω_2, then

$$\frac{A\,C}{C\,B} = \frac{\omega_2}{\omega_1}$$

and

$$\overline{A\,C} = \frac{\omega_2}{\omega_1 + \omega_2}\,A\,B, \quad \overline{C\,B} = \frac{\omega_1}{\omega_1 + \omega_2}\,\overline{A\,B}$$

Substituting $A\,Q = A\,C + C\,Q$, and $B\,Q = C\,Q - C\,B$ in (2), it is easily deduced that

$$v_1 = (\omega_1 + \omega_2)\,\overline{C\,Q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From (3) and (4) it is evident that the resultant velocity of P is $(\omega_1 + \omega_2)\,\overline{C\,P}$. That is, any point P, and therefore the whole body, is rotating with angular speed equal to the sum of the component angular speeds, about a parallel axis in the same plane, and distant from the axis of the component rotations inversely as the component angular speeds.

The above result can be more simply attained by an application of the principle of 'addition of vectors.' Let ρ be the vector $A\,P$, from the axis A to any point P of the rotating body, and let a be the vector $A\,B$. Then $P\,a$ is a vector of magnitude $\omega_1\,\rho$, at right angles to ρ; $B\,P$ is the vector $(\rho - a)$; and $P\,b$ is a vector of magnitude $\omega_2\,(\rho - a)$, at right angles to $(\rho - a)$.

$$\begin{aligned}
\text{Vector } Pc &= \text{vector } Pa + \text{vector } Pb \\
&= \omega_1 \rho + \omega_2 (\rho - a) \\
&= (\omega_1 + \omega_2)\rho - \omega_2 a \\
&= (\omega_1 + \omega_2)\left(\rho - \frac{\omega_2}{\omega_1 + \omega_2} a\right) \\
&= (\omega_1 + \omega_2)(\rho - A\,C) \\
&= (\omega_1 + \omega_2)\,C\,P, \text{ and at right angles to } C\,P.
\end{aligned}$$

That is, any point P rotates about the axis C (where $A\,C : C\,B = \omega_2 : \omega_1$) with angular speed equal to the sum of the component angular speeds.

Let figure 26 be a view of the body taken in a direction at right angles to that of figure 25, $A\,B$ now representing the plane of the motion. The rotation ω_1 may be represented by a line $A\,A_1$ at right angles to $A\,B$, its length representing, to some scale, the magnitude of the rotation ω_1. In the same way $B\,B_1$ may represent the rotation ω_2. The resultant rotation, $C\,C$, is equal to the sum of the rotations ω_1 and ω_2, and takes place about an axis whose distances from A and B are inversely proportional to the rotations ω_1 and ω_2.

Thus, rotations about parallel axes can be represented in the same way as parallel forces, and their resultant found by the methods used to find the resultant of parallel forces (*see* chapter vi.).

Example.—Find the instantaneous centre of rotation of the crank of a front-driver geared two to one. Let n be the number of revolutions the cranks make in a minute, the wheel makes $2\,n$ revolutions, and the crank must make n revolutions backward relative to the wheel—*i.e.* makes $-\,n$ revolutions per minute. The crank's motion may be considered as the resultant of a rotation $2\,n$ about B, the point of contact of the wheel with the ground, and a rotation $-\,n$ about the wheel centre A (fig. 27). Applying the preceding results, the instantaneous centre is on the line $A\,B$, and

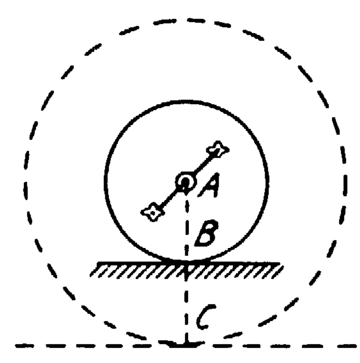


FIG. 27.

$$\therefore \quad \frac{AC}{CB} = \frac{2n}{-n} = -2.$$

That is, $\quad AC = -2\,CB$

or $\quad AC = 2\,BC = 2\,AB.$

The motion of the cranks relative to the ground is, therefore, the same as if they were fixed to a wheel twice the size of the driving-wheel, and running on a flat surface below the ground.

Translation and Rotation.—Let a body be subjected to a rotation ω_1 about an axis A (fig. 25), and to a translation with velocity v in a direction f_1f in the same plane as that of the motion. From A draw Af at right angles to ff_1. Let P be any point on the body. From P draw PQ at right angles to Af. Then proceeding as before, the components of P's resultant velocity at right angles and parallel to Af are respectively

$$v_1 = (\omega_1 \times AQ) - v \quad \ldots \ldots \quad (5)$$

$$v_2 = \omega_1 \times QP \quad \ldots \ldots \quad (6)$$

Let C be a point on Af such that $(\omega_1 \times AC) = v$; then (5) becomes

$$v_1 = \omega_1 \times (AQ - AC) = \omega_1 \times CQ \quad \ldots \quad (7)$$

By comparing (6) and (7), it is evident that the resultant velocity of P is one of rotation about the centre C with angular velocity ω_1. Thus, the resultant of a rotation and a translation is a rotation of the same magnitude about a parallel axis, the plane of the two axes being at right angles to the direction of translation.

Example.—A cycle wheel, relative to the frame, has a motion of rotation about the axle; the frame, and therefore the axle, has a motion of translation. The instantaneous motion of the wheel is the resultant of these two motions. The resultant axis of rotation of the wheel is the point of contact with the ground.

36. **Simple Cases of Relative Motion of Two Bodies in Contact.**—In the theory of bearings it is important to know the relative motion of the portions of two bodies in the immediate neighbourhood of the point of contact, the motion of the bodies

being such that they remain always in contact. Before discussing the general case we will notice a few simpler examples. It will be convenient to consider one of the bodies as fixed, we will then have to speak only of the motion of one of the bodies; this may be done without in any way altering the relative motion.

Sliding.—If the motion of the body can be expressed as a simple translation, 'sliding' is said to take place at the point of contact. With this definition, pure sliding can only exist continuously when the surface of either the fixed or moving body is cylindrical; the elements of the surfaces at the point of contact will constitute a 'sliding pair.' An example is afforded by the motion of a pump-plunger in its barrel.

Rolling.—If the instantaneous axis of rotation passes through, and lies in the tangent plane at, the point of contact of the fixed and moving bodies, the motion is said to be 'rolling'; the rolling is therefore the same as the relative rotation. At the point of contact of a wheel rolling along the ground, the motion is pure rolling. The position of the instantaneous axis continually changes; but in plane motion it always preserves the same direction.

Spinning.—If the instantaneous axis of rotation passes through, and it is at right angles to the tangent plane at, the point of contact, the motion is similar to the spinning of a top, and may be called *spinning.* An example of pure spinning is found at the centre of a pivot-bearing.

Rubbing.—In a turning pair, the motion can be expressed as a simple rotation about the axis of the pair. For example, the motion of a shaft of radius r in a plain cylindrical bearing is a rotation, ω, about the centre o of the bearing (fig. 28). The motion can also be expressed as an equal rotation, ω, about a parallel axis through P, a point on the surface of the bearing, and a translation with speed $v = \omega r$ in the direction PT at right angles to OP. The motion at P is kinematically more complex than 'sliding,' as above defined, and yet there is nothing of what is commonly understood as *rolling*; we may give it the name *rubbing.* Thus, rubbing at any point on the surface of contact of a cylindrical shaft of radius

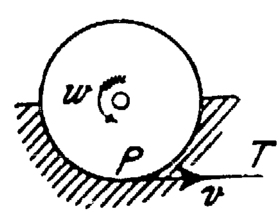


FIG. 28.

r is equivalent to a translation v and a rotation $\frac{v}{r}$ about an axis, parallel to that of the shaft, passing through the point in question.

More generally, let A and B be two bodies in contact at the point P (fig. 29), let r_a and r_b be their respective radii of curvature at P, and let I be the instantaneous axis of rotation of angular speed ω. I must lie on the common normal at P, since the bodies remain in contact during the motion. Suppose A fixed, and that the same point of the body B rubs along A with speed V for at least two consecutive instants. The motion of B on A may then be said to be pure rubbing. In this case I must

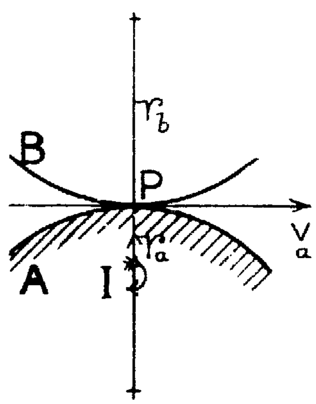


FIG. 29.

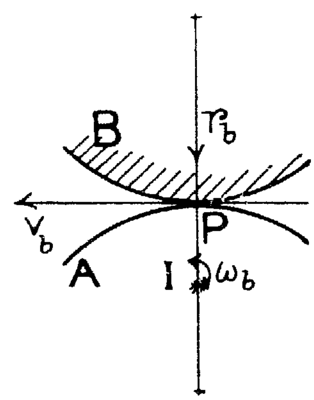


FIG. 30.

evidently coincide with the centre of curvature of the body A at the point P; then U_a, the rubbing of B on A, takes place with speed, $V_a = \omega r_a$, and is therefore equivalent to a translation V_a and a rotation $\frac{V_a}{r_a}$, or

$$U_a \equiv V_a \text{ and } \frac{V_a}{r_a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Similarly, if the position of I be such that the same point of A rubs on B for at least two consecutive instants,

$$U_b \equiv V_b \text{ and } \frac{V_b}{r_b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

37. **Combined Rolling and Rubbing.**—In figure 29 let A be fixed, and let the motion of the body B be kinematically a translation $V_a = V$, and a rotation $\omega_a = \omega$ about the point of contact P. The motion at P is compounded of rubbing and

rolling. The rubbing has already been defined; R_a, the rolling of B on A, will be the total motion less the rubbing, *i.e.*—

$$R_a = (V_a \text{ and } \omega_a) - \left(V_a \text{ and } \frac{V_a}{r_a}\right)$$

$$= \omega_a - \frac{V_a}{r_a} = \omega - \frac{V}{r_a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

In using the formula (10) the positive directions of the axis of ω_a, of r_a, and of V_a should be taken so that, in the order named, they form a right-handed system of rectangular axes. That is, looking along the positive direction of the axis of ω, a positive rotation, ω, will appear clock-wise, and the positive direction of r if rotated a right angle in the positive direction of ω, will come into the positive direction of V. r_a and r_b may be taken positive for convex surfaces, negative for concave surfaces. The positive directions of ω_a, r_a and V_a are shown in figure 29.

In figure 30 let the relative motion of the bodies be exactly the same as in figure 29, but let B be fixed. Then V_b and ω_b will be oppositely directed in space to V_a and ω_a respectively. But with the above conventions as to positive directions, taking r_b positive, V_b will be positive and equal to V, ω_b will be negative and equal to $-\omega$. Therefore

$$R_b = \omega_b - \frac{V_b}{r_b} = -\omega - \frac{V}{r_b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

From formulæ (10) and (11) it is seen that when rolling and rubbing combined take place, the 'rollings' of the two bodies are not reciprocal. The actions at the points of contact in the two bodies are not reciprocal, as may be illustrated by a few examples.

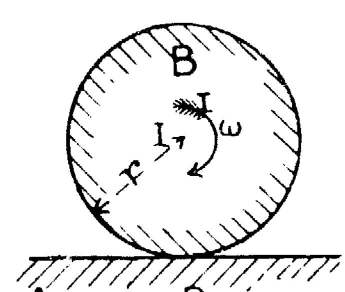


FIG. 31.

Example I.—Let the bodies A and B be a plane and cylinder of radius r respectively, in contact at P (fig. 31). Let the instantaneous axis of rotation coincide with the axis of the cylinder, and let ω be the angular speed of B relative to A. Then at P:—$r_a = \infty$; $r_b = r$; the speed of rubbing $V = V_a, = -\omega r$.

$$R_a = \omega - \frac{V}{r_a} = \omega$$

$$R_b = -\omega - \frac{V}{r_b} = -\omega - \frac{-\omega r}{r} = 0.$$

That is, the cylinder's motion on the plane is compounded of a rubbing of speed ωr, and a rolling of angular speed ω. The plane's motion on the cylinder is one of pure rubbing with speed ωr.

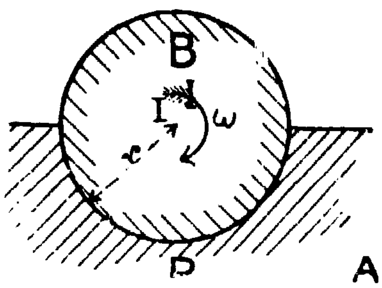


FIG. 32.

Example II.—Let the bodies A and B be a circular bearing and shaft respectively, of the same radius r (fig. 32), ω being the angular speed of the shaft. Then at P, $r_a = -r$, $r_b = r$, $V = V_a = -\omega r$, and

$$R_a = \omega - \frac{V}{r_a} = \omega - \frac{-\omega r}{-r} = 0$$

$$R_b = -\omega - \frac{V}{r_b} = -\omega - \frac{-\omega r}{r} = 0.$$

Thus the definitions given in (10) and (11) of the magnitudes of the rollings of one body on the other are consistent with our usual conceptions in these simple cases.

CHAPTER V

KINEMATICS: MOTION IN THREE DIMENSIONS

38. **Resultant of Translations.**—If a body be subjected to a number of translations in different directions in space, the resultant velocity can be found by finding the resultant of any two of the given translations, which resultant must evidently lie in the same plane as the two given translations. The resultant of a third given translation with the resultant of the first two can again be found by the same method; and so on for any number of given translations. Thus the resultant of any number of translations in space is a motion of translation.

39. **Resultant of Two Rotations about Intersecting Axes.**—Let the axes $O\,A$ and $O\,B$ of the rotations intersect at the point O (fig. 33). The rotations ω_1 and ω_2 may be represented by the length of the lines $O\,A$ and $O\,B$ respectively, and since rotations are resolved and compounded like forces, the resultant rotation will be represented by the diagonal $O\,C$ of the parallelogram of which $O\,A$ and $O\,B$ are adjacent sides. This proposition

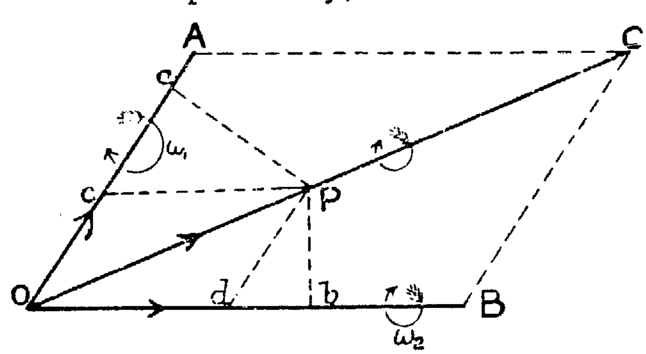


FIG. 33.

is called the *parallelogram of rotations*. In using this proposition, attention must be paid to the sense of the rotation. The lengths of the lines representing the magnitudes of the rotations must be set off along the axes of the rotations in such directions that when looking in the positive directions the motions both appear either in watch-hand direction, or both in contra watch-hand direction. In figure 33, the rotations are both in watch-hand direction; the

resultant rotation about the axis $O\,C$ will therefore be in the direction indicated by the arrow.

The above proposition is so important that a separate proof depending on first principles will be instructive. Let $O\,A$ and $O\,B$ be the axes of rotation, and let P be a point on the body lying in the plane $A\,O\,B$. Draw $P\,a$ and $P\,b$ perpendicular to $O\,A$ and $O\,B$ respectively. If P lie in the angle between the positive directions $O\,A$ and $O\,B$, the linear velocity of P, which is in a direction at right angles to the plane of the axes, will be

$$\omega_1\,\overline{a\,P} - \omega_2\,b\,P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

If P lie on the axis of resultant rotation its velocity is zero, and (3) becomes $\omega_1\,a\,P - \omega_2\,\overline{b\,P} = O$,

or,
$$\frac{\omega_1}{\omega_2} = \frac{b\,P}{a\,P}$$

Draw $P\,c$ and $P\,d$ parallel respectively to $O\,B$ and $O\,A$, meeting $O\,A$ and $O\,B$ at c and d respectively. Then, the triangles $P\,a\,c$ and $P\,b\,d$ are similar, and therefore—

$$\frac{b\,P}{a\,P} = \frac{P\,d}{P\,c} = \frac{O\,c}{O\,d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

That is, $O\,P$ is the diagonal of a parallelogram whose adjacent sides coincide with the direction of the axis of rotation, and are of lengths respectively proportional to the component angular velocities about these axes.

4c. **Resultant of Two Rotations about Non-intersecting Axes.**—Let $A\,A$ and $B\,B$ (fig. 34) be the two axes, and let $g\,h$ be the common perpendicular to $A\,A$ and $B\,B$. Through h draw a line $C\,C$ parallel to $A\,A$. Then by section 35, the rotation ω_1 about the axis $A\,A$ is equivalent to an equal rotation about the axis $C\,C$,

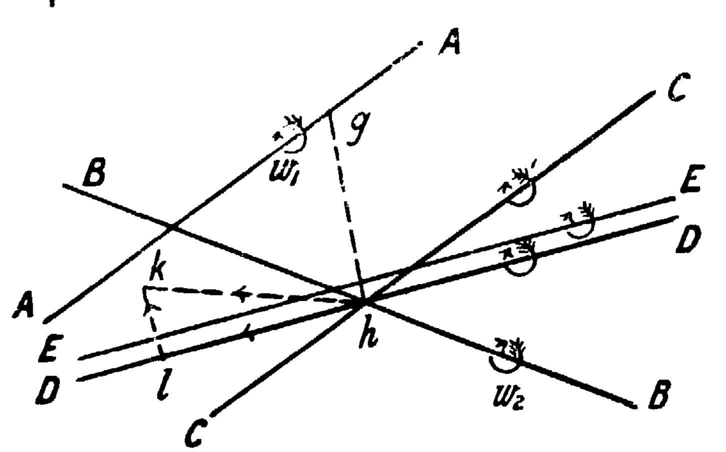


FIG. 34.

together with a translation in the direction $h\,k$ at right angles to the plane containing $A\,A$ and $C\,C$. The resultant of the rotations about the axes $B\,B$ and $C\,C$ is, by section 39, a rotation about an axis $D\,D$ passing through h. Thus, the given motion is equivalent to a rotation about an axis $D\,D$, and a translation in the direction $h\,k$. The translation in the direction $h\,k$ may be resolved into two components, $h\,l$ along $D\,D$ and $l\,k$ at right angles to $D\,D$. By section 35, the rotation about $D\,D$ and the translation in the direction $l\,k$ are equivalent to an equal rotation about a parallel axis $E\,E$. Thus, finally, the resultant motion is a rotation about an axis $E\,E$ and a translation in the direction of that axis. Such a motion is called a *screw* motion.

41. **Most General Motion of a Rigid Body.** In the same way it can be shown that the resultant of any number of translations and of any number of rotations about intersecting or non-intersecting axes may be reduced to a rotation about an axis and a translation in the direction of that axis. If a common screw bolt be fixed and its nut be moved, the motion imparted is of this character. The motion of the nut can be specified by giving the *pitch* of the screw and its angular speed of rotation about its axis. In the same way, the motion of a rigid body at any instant can be expressed by specifying the axis and pitch of its screw, and its angular speed.

42. **Most General Motion of Two Bodies in Contact.** We have seen that the most general motion of a rigid body can be resolved into a rotation ω and a translation v in the direction of the axis of rotation. Also that a rotation about any axis is equivalent to an equal rotation about a parallel axis through any point, together with a translation at right angles to the plane of the parallel axes. Hence, if two bodies move in contact, the relative motion at any point of contact can be resolved into a translation, and a rotation about an axis passing through the point of contact. The direction of the translation must be in the tangent plane at the point; since, if the two bodies move in contact, there can be no component of the translation in the direction of t[illegible]rmal.

Let figure 35 be a section of the two bodies A and B by a plane, passing through the point of contact P, at right angles to

the instantaneous axis of rotation II. The body A may be considered fixed, the body B to have a rotation ω round, and a translation v along, II. If PI be perpendicular to II, the motion of B is equivalent to a rotation ω about the axis Pa, parallel to II, together with a translation $\omega \cdot IP$ along Pb at right angles to the plane of PI and Pa, plus a translation v along Pa. The resultant of these two translations is a translation V along Pc. Pc must lie in the common tangent plane to the surfaces at P.

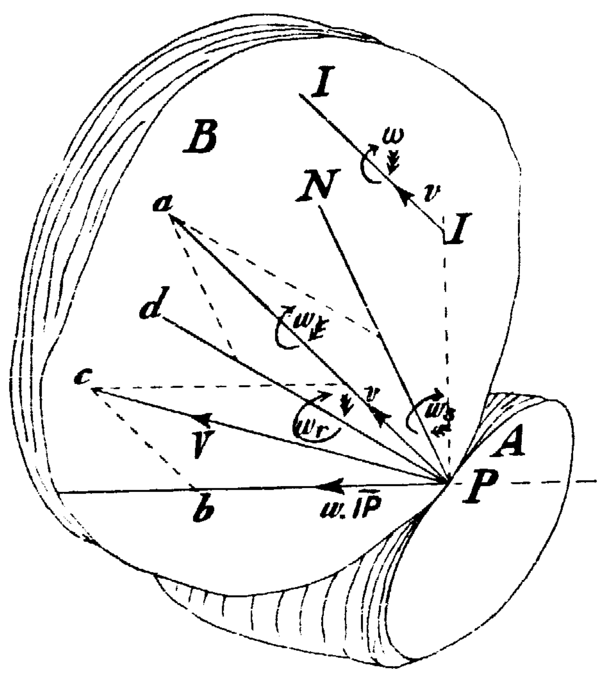


FIG. 35.

Let PN be the normal at P, and Pd the intersection of the tangent plane with the plane containing PN and Pa. Then, the rotation ω about Pa can be resolved into rotations ω_s and ω_r about PN and Pd respectively. Thus, the motion at P consists of translation with velocity V in the direction Pc, a spinning, ω_s, about the normal PN, and a rolling, ω_r, about the axis Pd lying in the tangent plane. Therefore the most general relative motion of two bodies in contact is compounded of 'rubbing,' 'rolling,' and 'spinning.'

We have in the chapter on Plane Motion given examples of the pure motions just mentioned. We shall see, in the chapter on Bearings, that the motion of a ball on its path in the ordinary form of adjustable bearing is compounded of rolling and spinning; while, in some special ball-bearings, the motion at the point of contact of a ball with its path is compounded of rubbing, rolling, and spinning.

CHAPTER VI

STATICS

43. **Graphic Representation of Force.**—For the complete specification of a force acting on a body, its direction, line of application, and magnitude are required. A force can therefore be represented completely by a straight line drawn on a diagram, the length of the line representing to scale its magnitude, the direction and position of the line giving the direction and positions of application of the force. Thus a force can be represented by a *localised* vector.

44. **Parallelogram of Forces.**—When two or more forces are applied at the same point, a single force can be found which is equivalent to the original forces. This is called the *resultant* force, and the original forces are called the components. If the forces act in the same direction, the resultant is, of course, equal to the sum of the component forces. If two forces act in opposite directions, the resultant is the difference of the two. Generally, if a number of forces act along a straight line, some in one direction, others in the opposite direction, the resultant of the whole system is equal to the difference between the sum of the forces acting in one direction and that of the forces acting in the opposite direction.

Suppose two forces acting at a point in different directions are represented by $o\,a$ and $o\,b$ respectively (fig. 36), then it is evident that some force such as $o\,c$ in a direction between $o\,a$ and $o\,b$ will be the resultant. The resultant $o\,c$ is found by completing the parallelogram $o\,a\,c\,b$ and drawing the diagonal $a\,c$, exactly as in the case of the parallelogram of velocities.

Want of space prevents a strict elementary mathematical proof of this proposition, but it can be easily verified experi-

mentally as follows: Fasten two pulleys, A and B (fig. 37), on a wall, the pulleys turning with as little friction as possible on their spindles. Take three cords jointed together at O with weights W_1, W_2, W_3 at their ends. Let the heaviest weight hang vertically downwards from O, and let the other two cords be passed over the pulleys A and B respectively. Then, if the heaviest weight, W_3, underneath O be less than the sum of the other two, the whole system will come to rest in some particular

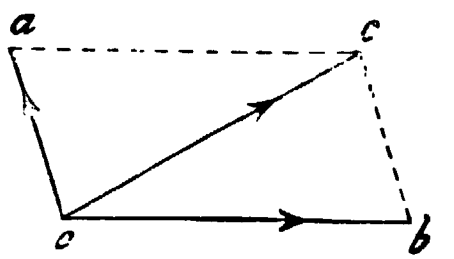


FIG. 36.

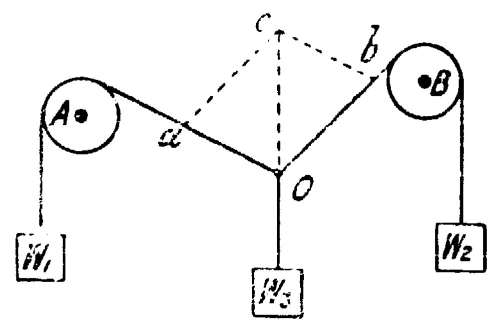


FIG. 37.

position. While in this position make a drawing on the wall of the three cords meeting at o. Produce the vertical cord upwards to any point c, and from c draw parallels $c\,a$ and $c\,b$ to the other two cords. It will be found on measurement that the lengths $O\,a$, $O\,b$, and $O\,c$ are exactly proportional to the weights W_1, W_2, and W_3. Thus the resultant of the forces along $O\,a$ and $O\,b$ is given by the diagonal $O\,c$ of the parallelogram whose sides represent the component forces.

Example.—The crank spindle of a bicycle is pressed vertically downwards by the rider with a force of 25 lbs., while the horizontal pull of the chain is 50 lbs. What is the magnitude and direction of the resultant pressure on the bearing? Set off $O\,A$ (fig. 38) vertically downwards equal to 25 lbs. and $O\,B$ horizontally equal to 50 lbs. Complete the parallelogram $O\,A\,B\,C$. The resultant is equal in magnitude and direction to the diagonal $O\,C$, which by measurement is found to be 55·9 lbs.

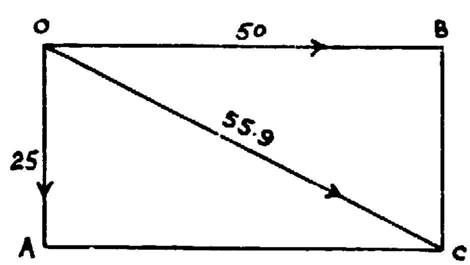


FIG. 38.

45. **Triangle of Forces.**—Suppose that in addition to the two forces $o\,a$ and $o\,b$ (fig. 36) a third force, $c\,o$, acts at the point;

this third force being exactly equal, but opposite to, the resultant of the two forces. If these three forces act simultaneously no effect will be produced, and the body will remain at rest. bc is equal and parallel to oa, and may therefore represent in magnitude and direction the force oa acting at A. The three sides ob, bc, and co of the triangle obc, therefore, taken in order, represent the three forces acting at the point and producing equilibrium. The proposition of the parallelogram of forces may therefore be put in the following form, which is often convenient:

If three forces act at a point and produce equilibrium they can be represented in magnitude and direction by the three sides of a triangle taken in order round the triangle. The converse of this proposition is also true.

A very important proposition which can be deduced immediately from the triangle of forces is, that if three forces act on a body and produce equilibrium they must all act through the same point.

46. **Polygon of Forces.**—Since forces acting at a point can be represented by vectors, the resultant R of a number of forces,

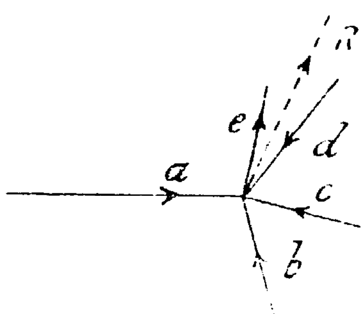


FIG. 39.

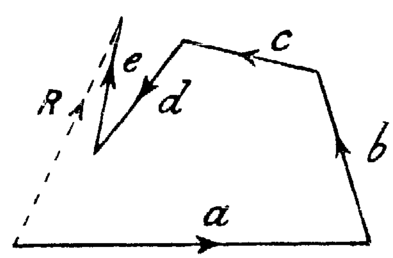


FIG. 40.

a, b, c, d, and e, acting at the same point (fig. 39) can be found by drawing a vector polygon (fig. 40) whose sides represent the given forces; the resultant vector R represents the resultant force. If a force equal, but oppositely directed, to R acted at the same point as the forces a, b, c, d, and e, they would be in equilibrium. Therefore, if a number of forces acting at a point are in equilibrium, they can be represented in magnitude and direction by the sides of a polygon, taken in order round the polygon. Conversely, if a number of forces acting at a point

are represented in magnitude and direction by the sides of a polygon taken in order, they are in equilibrium.

In the preceding paragraph it must be clearly understood that the sides of the polygon represent the forces in magnitude and direction, but not in *position*. Thus the sides of the polygon a, b, c, d, e (fig. 40) represent in magnitude and direction the five forces acting at the same point. If a body were acted on by forces represented by the sides of a polygon, in *position* as well as in magnitude and direction, a turning motion would evidently be imparted to it.

47. **Resultant of any Number of Co-planar Forces.**—The resultant of any number of forces all lying in the same plane acting on a rigid body, and which do not necessarily all act at the same point, may be found by repeated applications of the principle of the parallelogram of forces. The resultant R_2 of any two of the given forces P_1 and P_2 passes through the point of intersection of the latter; the resultant R_3 of R_2 and a third force, P_3, passes through the point of intersection of R_2 and P_3; and so on. This process is very tedious when a great number of forces have to be dealt with. The following method is more convenient:

Let figure 41 represent the position of the given forces, and figure 42 the corresponding force-polygon $P_1 P_2 \ldots$ The resultant R of all the given forces is evidently represented in magnitude and direction by the line af forming the closing side of the polygon; for if a force of magnitude and direction fa were added to the given forces, the resultant would be of zero magnitude. It only remains therefore to determine the *position* of the resultant R on figure 41.

No difference will be made if two equal and opposite forces be added to the system. We will add a force Q, represented by Oa in the force-polygon, which acts along any line a (fig. 41). The resultant of Q and P_1 is Ob (fig. 42), and it passes through p_1, the point of intersection of Q and P_1 (fig. 41). Draw from the point p_1 the line b parallel to Ob (fig. 42), cutting the line of action of P_2 at p_2. The resultant of Q, P_1, and P_2 is Oc (fig. 42), and it passes through p_2. Draw from the point p_2 the line c parallel to Oc (fig. 42). Continuing this process, the resultant

of Q, P_1, P_2, P_3, P_4, and P_5 is Of (fig. 42), and acts through the point p_5. From p_5 draw the line f parallel to Of (fig. 42),

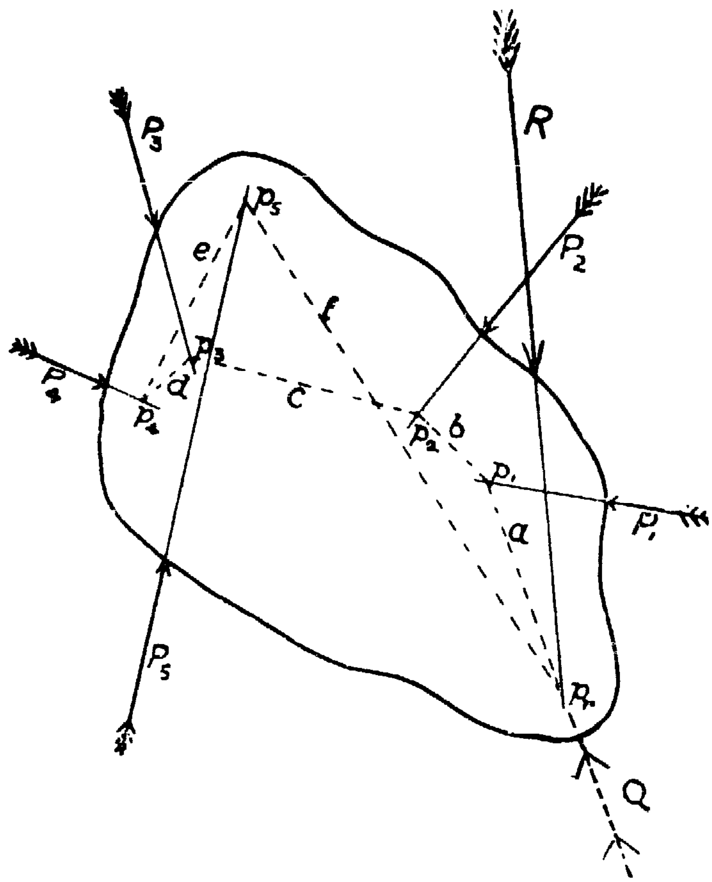


FIG. 41.

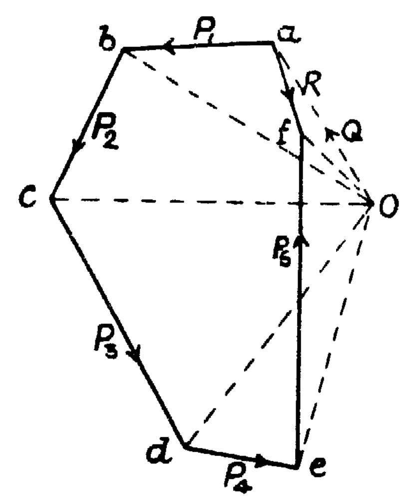


FIG. 42.

cutting the line a, the line of action of the added force Q, at p_r. The resultant of Of and $-Q$ is $af = R$ (fig. 42), and it acts through the point p_r.

The above construction may be expressed thus: Take any pole O and from it draw radius vectors to the corners of the force-polygon. Draw another polygon, which may be called the *link-polygon*, having its corners p_1, p_2 . . . on the lines of action of the given forces P_1, P_2, . . . and having its sides a, b, . . . parallel to the radius vectors $O\,a$, $O\,b$ . . . of the force-polygon; the sequence of sides and corners a, p_1, b, p_2 . . . in the link-polygon being the same as that of the corners and sides a, P_1, b, P_2, . . . of the force-polygon. The point of intersection of the first and last sides of the link-polygon determines the position of the resultant R.

It is readily seen from the above, that if a system of forces acting on a rigid body are in equilibrium, both the force- and link-polygons must be closed.

48. **Resolution of Forces.**— A single force may be resolved into two components in given lines which intersect on the line of action of the given force. The principle of the parallelogram of forces is, of course, used again here. Let $o\,c$ (fig. 43) be the given force acting at o, and let its components in the directions $o\,a$ and $o\,b$ be required. From c draw $c\,a$ and $c\,b$ respectively parallel to $b\,o$ and $a\,o$, meeting $o\,a$ and $o\,b$ in a and b respectively: $o\,a$ and $o\,b$ are the required components of the given force in the two given directions.

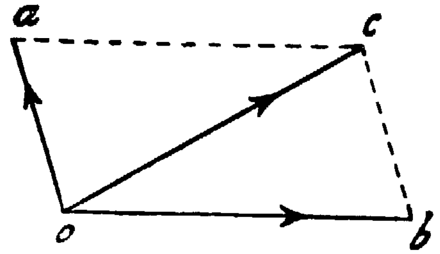


FIG. 43.

Example.—Given the vertical pressure on the hub of the driving-wheel of a Safety bicycle, to find the forces acting along the top and bottom forks, $O\,A$ and $O\,B$ (fig. 44).

Draw $O\,c$ vertical and equal to the given pressure on the hub. This is the direction and magnitude of the force with which the wheel presses on the hub spindle. From c draw $c\,a$ and $c\,b$ parallel to $O\,B$ and $A\,O$ respectively, meeting $O\,A$ and $B\,O$ produced in a and b respectively. $o\,a$ and $o\,b$ are the forces acting along the top and bottom forks respectively. It will be seen that the top fork $O\,A$ is compressed and the bottom fork $O\,B$ is in tension.

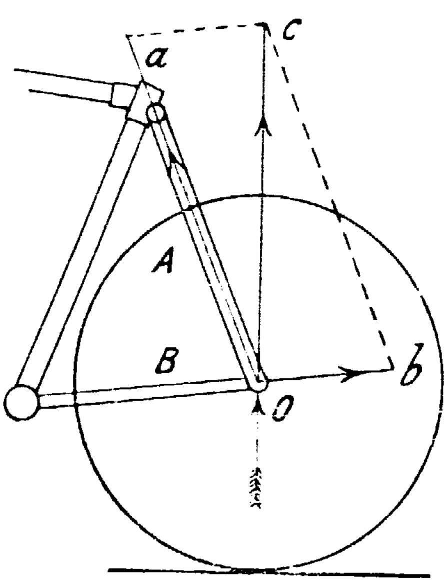


FIG. 44.

Resolution of a Force into Three Components in given Directions and Positions.—Let R be a force whose components acting along the given lines P_1, P_2, and P_3 (fig. 45) are required. Let R and P_1 intersect at A, P_2 and P_3 intersect at B. Then R may be resolved into two forces acting along P_1 and $A\,B$ respectively, the latter into two forces acting along P_2 and P_3 respectively. The constructions necessary are indicated in fig. 46.

Any force, R, acting on a rigid body can be resolved into two, one acting along a given line P_1, the other passing through a given point B. The latter force must pass through A, the point

of intersection of R and P_1. The construction is clearly shown in figures 45 and 46.

If the point of intersection A be inaccessible, as in figure 47, the link-polygon method may be used with advantage. In the force diagram (fig. 48) set off $a\,f$ equal to R to any convenient scale, draw $f\,b$ parallel to P_1. Commence the link-polygon at B, by drawing the side a parallel to the vector $O\,a$, then draw the side f parallel to the vector $O\,f$, cutting the line of action of P_1 at p_1. The closing side b of the link-polygon is the straight line $p_1\ B$. Draw the vector $O\,b$ parallel to the side b of the link-polygon, cutting the side P_1 of the force triangle at b. The force P_2 is represented in magnitude and direction by the

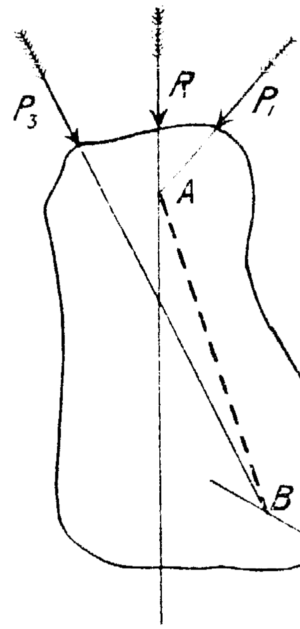


FIG. 45.

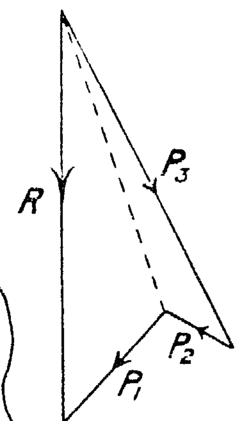


FIG. 46.

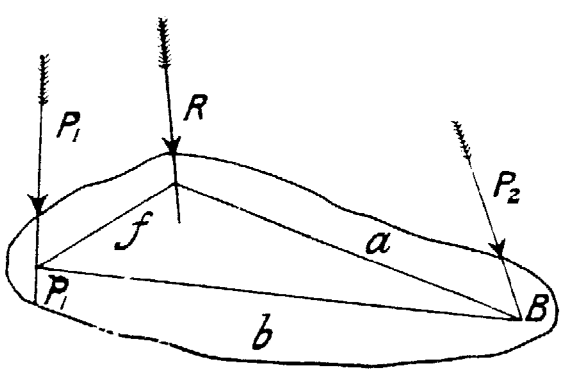


FIG. 47.

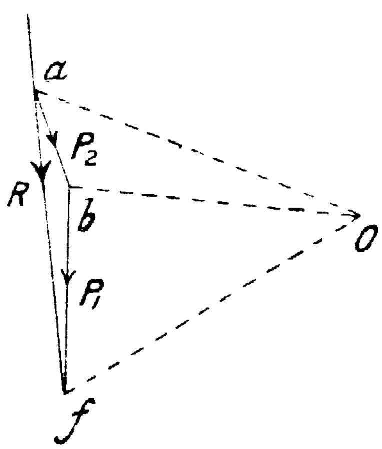


FIG. 48.

third side $a\,b$ of the force triangle. Comparing with figures 41 and 42, the truth of the above construction is obvious.

49. **Parallel Forces.**—Let two parallel forces P_1 and P_2 act on a body (fig. 49). Required to find their resultant. It is evident that the resultant force R is equal to the sum $P_1 + P_2$; the only element to be found is the point at which it acts. Let $A\,B$ be a line in the body at right angles to the directions of P_1 and P_2, and let C be the point at which the resultant R acts.

Let another force, Q, equal and opposite to R, be applied to the body; then since it is equal and opposite to the resultant of P_1 and P_2, the body is in equilibrium under the action of the three forces P_1, P_2, and Q. Consider the moments of the forces about the point C; that of Q is zero, and, therefore, the algebraic sum of the moments of P_1 and P_2 must also be zero, since the body is in equilibrium. Therefore,

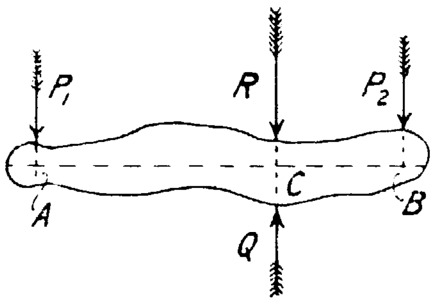


FIG. 49.

$$P_2 \times CB = P_1 \times AC \quad . \quad (1)$$

that is, the point C divides AB into two parts inversely proportionate to the forces P_1 and P_2.

If the forces P_1 and P_2 acted in opposite directions (fig. 50), paying attention to the sign of the moments, it is seen that the point C will lie beyond A, the point of application of the larger force. Here again

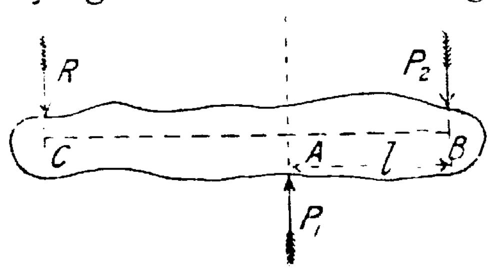


FIG. 50.

$$P_2 \times \overline{CB} = P_1 \times \overline{AC} \quad . \quad (1)$$

The above is often referred to as the principle of the lever. The experimental verification is easy.

The resultant of any number of parallel forces P_1, P_2, ... can be found by the method of figures 41 and 42; the force-polygon (fig. 42) becoming in this case a straight line.

50. **Mass-centre.**—An important case of finding the resultant of a number of parallel forces is finding the centre of gravity of a body. The earth exerts an attraction on every part of the body, and therefore the resultant force of gravity on the body is the resultant of a great number of parallel forces.

Considering a body as made up of an indefinite number of small particles of equal mass, the *mass-centre* of the body is a point such that its distance from any plane is the mean distance of all the particles from that plane. If the body is subjected to gravitational attraction, every particle is acted on by a force, the total force acting on the body is the resultant of all such forces. The centre of gravity is a point at which the total mass of the body may be considered to be concentrated, in considering its

attraction by other bodies. When the attractions on the particles of a body are proportional to their mass, as is practically the case on the surface of the earth, the mass-centre and the centre of gravity of a body are coincident.

If the density of the body is uniform, the mass centre will also be the geometrical centre of figure; in fact, it is the geometrical centre of figure that is of importance in problems on mechanics.

The mass-centres for a few important cases may be given here.

Circular, Square, or Rectangular Disc.—If these discs be cut out of metal plate of uniform thickness, it is evident that the mass-centre will also be at the geometrical centre of the figure.

Triangle.—Let $A\,B\,C$ (fig. 51) be a triangle, which we may consider cut out of thin metal plate. Consider any narrow strip, $p\,p$, parallel to the side $B\,C$; the mass-centre p_1 of this strip is at the middle of its length. Dividing up the triangle into a number of such slips, their mass-centres will all lie on the line $A\,a$, joining A to the middle point of $B\,C$. In the same way, by dividing the triangle up into a number of strips parallel to $A\,B$, it may be seen that the mass-centres of all the strips will lie on the line $C\,c$ joining C, the middle point of $A\,B$. The mass-centre of the whole triangle must lie somewhere on the line $A\,a$; it must also lie somewhere on the line $C\,c$; O, the point of intersection of these lines, is therefore the mass-centre. It can easily be proved that $a\,O$ is one-third of $a\,A$, and $C\,o$ one-third of $c\,C$.

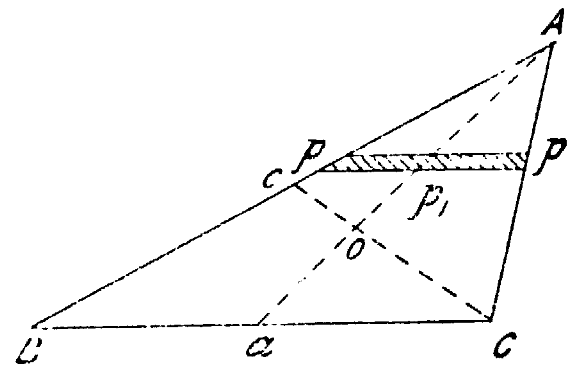


FIG. 51.

Circular Arc.—Let $A\,B$ (fig. 52) be a portion of a circular arc with centre O. Consider the moment about any diameter $O\,X$. Let $P\,P^1$ be a portion of the arc so short that it does not sensibly differ from the straight line $P\,P^1$, and its length is negligible in comparison with the radius. The mass may be

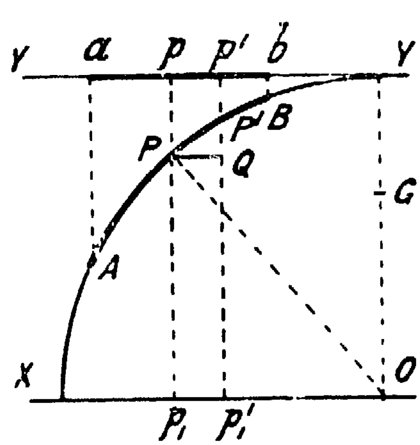


FIG. 52.

considered proportional to the length of the line, and we may therefore say that the moment of $P P^1$ about $O X$ is $P P^1 \times P p_1$; $P p_1$ being drawn perpendicular to $O X$; and $P^1 Q$ being negligible compared with $P p_1$.

Draw $Y Y$ a tangent to the circle and parallel to the axis $O X$; from A, P, P^1 and B project a, p, p^1 and b on this tangent, the projectors being at right angles to it. Draw $P Q$ parallel to, and $P^1 Q$ at right angles to $O X$, the two lines meeting at Q. Join $O P$. Then, since the triangles $P P^1 Q$ and $O P p_1$ are similar,

$$\frac{P P^1}{P Q} = \frac{P O}{P p_1}.$$

Therefore, $P P^1 \times P p_1 = P Q \times P O = p p^1 \times p p_1$ *i.e.* the moment of the arc $P P^1$ about the axis $O X$ is equal to the moment of the straight line $p p^1$ about the same axis.

This holds for all the elements of which the arc $A B$ may be considered made up. Therefore, by summing the moments of these elements we get the important result, that the moment of the arc $A B$ about the axis $O X$ is equal to the moment of the straight line $a b$, its projection on the tangent parallel to the axis.

If the arc under consideration be a semicircle of radius r, and G be its mass-centre, its length is πr, the length of its projection on the tangent is $2 r$, and we get

$$\pi r \times O G = 2 r \times r.$$

Therefore $$O G = \frac{2}{\pi} r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

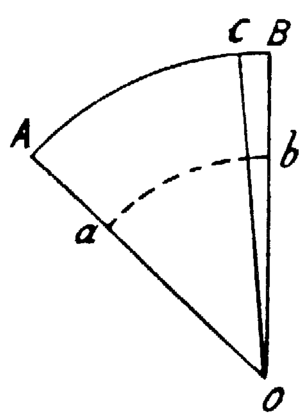


FIG. 53

Sector of a Circle. The mass-centre of a sector of a circle $O A B$ (fig. 53) is found by dividing it up into a number of smaller sectors, $O C B$, the arc $B C$ being so short as not to differ sensibly from a straight line. The sector $O C B$ may then be considered a triangle, its mass-centre will be at a distance from O equal to two-thirds $O B$. Thus, the mass-centres of the small sectors into which $O A B$ can be divided all lie on the arc $a b$, whose radius is two-thirds that of the arc

$A\,B$; and therefore the mass-centre of the sector $O\,A\,B$ is the same as that of the arc $a\,b$.

In particular, the centre of area included between a semi-circle and its diameter is at a distance $\frac{4}{3}\frac{r}{\pi}$ from the centre of the circle.

51. **Couples.**—If two parallel but opposite forces, P_1 and P_2 (fig. 54), are also equal, their resultant is zero, they tend to turn the body without giving it a motion of translation. Two equal, parallel, but oppositely directed forces constitute a *couple*, whose magnitude is measured by the product $P\,l$ of one of the equal forces into the perpendicular distance between their lines of action. A couple may be regarded as equivalent to a zero force acting at an infinite distance; with this point of view they form no exception to the general case of finding the resultant of given forces.

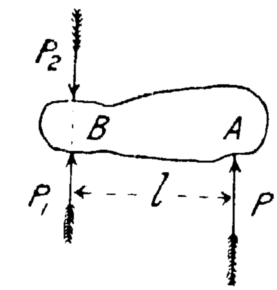


FIG. 54.

In the construction of figures 41 and 42, if the points a and f of the force-polygon coincide, the resultant of the given forces is zero. If, in addition, the line $p_5\,p_1$ is parallel to $O\,a$, the link-polygon is also closed, and the given forces are in equilibrium. If, however, $p_5\,p_1$ is not parallel to $O\,a$, the resultant of the given forces is a couple.

Let two parallel forces P_1 and P_2 (fig. 55), each equal to P, at a distance l apart, constitute a couple. The sum of the moments of the two forces about any point O in the plane of P_1 and P_2, distant x from P_1, is

$$P_2\,(x + l) - P_1\,x = P\,l;$$

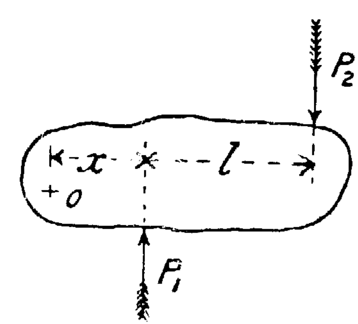


FIG. 55.

that is, the turning effect of a couple depends only on its moment $P\,l$, and not on the position of its constituent forces relative to the axis of turning. The axis of the couple is at right angles to its plane.

Let a single force P act on a body at A (fig. 54). Introduce at B two opposite forces P_1 and P_2, each equal to, and distant l from, P. No change in the condition of the body is effected by

this procedure, since P_1 and P_2 neutralise each other. But the system of forces may now be expressed as a single force P_1 acting at B, together with a couple Pl formed by the forces P and P_2. Thus, a force acting on a body at A is equivalent to an equal force acting at B, together with a *couple of transference* Pl.

A couple may be graphically represented by a vector parallel to its axis—*i.e.* at right angles to its plane; the length of the vector being equal, to some scale, to the moment Pl of the couple.

52. **Stable, Unstable, and Neutral Equilibrium.**—If a heavy body be situated so that a vertical line through its mass-centre passes within its base it is in equilibrium. If the vertical line through the mass-centre fall outside the base, the body is not in equilibrium, and will fall unless otherwise supported. If a body, supported in such a way that it is free to turn about an axis O (fig. 56), be left to itself it will come to rest in such a position that its mass-centre G will be vertically underneath the axis of suspension O. If the body be displaced slightly, so that its mass-centre is moved to G^1, when left to itself it will return to its original position. In fact, the forces now acting on the body are, its weight acting downwards through G^1, and the reaction at the support O acting vertically; these two forces form a couple evidently tending to restore the body to its original position. In this case the body is said to be in *stable* equilibrium.

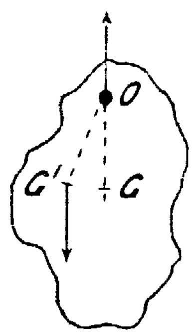


FIG. 56.

If now the body be placed with its mass-centre above O (fig. 57), though in equilibrium, the smallest displacement will move G sideways, and the body will fall. The equilibrium in this case is said to be *unstable*.

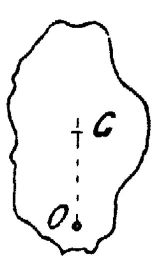


FIG. 57.

If the mass-centre of the body coincide with the axis of suspension, the body will remain at rest in any position, and the equilibrium in this case is said to be *neutral*.

A body may have equilibrium of one kind in one direction, and of another kind in another direction: thus a bicycle resting on the ground in its usual position is in stable equilibrium in a longitudinal direction, and is in unstable equilibrium in a trans-

verse direction. A bicycle wheel resting on the ground is in neutral equilibrium in a longitudinal direction, and in unstable equilibrium in a transverse direction.

53. **Resultant of any System of Forces.**—*Concurrent forces.*—If the given forces all pass through the same point, but do not all lie in the same plane, the method of section 46 can be extended to them; their resultant will be represented as before, by the closing side of the vector-polygon, the only difference from the case of coplanar forces being that the vector-polygon is no longer plane. Thus, the resultant of a system of concurrent forces is either zero or a single concurrent force.

Non-concurrent, non-planar forces.—Let P_1, P_2, . . . be the given system of forces. Take any point O as origin and introduce two opposite forces, p_1 and $-p_1$, each equal and parallel to P_1. No change is made by this procedure, since p_1 and $-p_1$ neutralise each other. The force P_1 is therefore equivalent to a single force p_1 acting at O, and a couple of transference $P_1 l_1$; l_1 being the length of the perpendicular from O to P_1, and the axis of the couple being perpendicular to the plane of P_1 and p_1. Similarly, P_2 is equivalent to an equal and parallel force p_2 acting at O, together with a couple of transference $P_2 l_2$; and so on for all the given forces. The resultant of the concurrent forces p_1, p_2 . . . is either zero or a single concurrent force, p. Since the couples $P_1 l_1$, $P_2 l_2$, . . . are vector quantities, their resultant is also a similar vector quantity—*i.e.* a couple C. Hence the resultant of any system of forces can be expressed as the sum of a single force p and a couple C.

The magnitude of p does not depend on the position of the origin O, while that of C does. The couple C can be resolved into two couples C' and C'', having their axes respectively parallel to, and at right angles to, the direction of p. The resultant of p and C'' is a force p', equal to, parallel to, and at a distance $\dfrac{C''}{p}$ in a direction at right angles to the plane of p and the axis C'' from, p. Thus, finally, the resultant of any system of forces can be expressed as a single force p' and a couple C'' having its axis parallel to p'.

CHAPTER VII

DYNAMICS—GENERAL PRINCIPLES

54. **Laws of Motion.**—In section 13 we have seen that the measurement of force is closely associated with that of motion. The general phenomena of force and motion have been summed up by Newton in his well-known laws of motion :

I. Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it may be compelled by applied forces to change that state.

II. Change of motion is proportional to the force applied, and takes place in the direction in which the force acts.

III. The mutual actions of any two bodies are always equal and oppositely directed in the same straight line ; or, action and reaction are equal and opposite.

These laws apply to forces acting in the direction of the motion, and also to forces acting in any other direction. A force like the latter will alter the direction of the body's motion, and may, or may not, increase or diminish its speed. It follows from Newton's first law that any body moving in a curved path must be continually acted on by some force so long as its motion in the curved path continues.

55. **Centrifugal Force.**—An important case of motion, especially to engineers and mechanicians, is uniform motion in a circle. If a stone at the end of a string be whirled round by hand, the string is drawn tight and a pull is exerted on the hand. This pull is called *centrifugal* force. At the other end the string exerts a pull on the stone tending to pull it inwards towards the hand. This pull is called the *centripetal* force, and it is the continual exercise of this force that gives the stone its circular path. If this force ceased to act at any instant the stone would continue

its motion, neglecting the influence of gravity, in a straight line in the direction it had at the instant the centripetal force ceased to act.

The distinction between the two forces must be carefully kept in mind.

Every point on the rim of a rapidly rotating bicycle wheel is acted on by a centripetal force which is supplied partially by the tension of the spokes. If the speed of rotation gets abnormally high, the centripetal force required to give the particles in the rim their curvilinear motion may be so great that the strength of the material is insufficient to transmit it, and the wheel bursts. The flywheels of steam engines are often run so near the speed limited by these considerations, that it is not uncommon for them to burst under the action of the centrifugal stress.

Let m be the mass in lbs. of the body moving with speed v feet per second in a circle of radius r. It has been shown (sec. 26) that the radial acceleration a is $\frac{v^2}{r}$. But if f be the radial force acting, by section 14,

$$f = ma = \frac{m v^2}{r} \text{ poundals, or } f = \frac{m v^2}{g r} \text{ lbs.} \quad . \quad . \quad . \quad (1)$$

56. **Work.**—When a force acts on a body and produces motion it is said to do *work*. If a force acts on a body at rest, and no motion is produced, no work is done. The idea of *motion* is essential to work. If a man support a load without moving it, although he may become greatly fatigued, he cannot be said to have done mechanical work. The load, as regards its mechanical surroundings, might as well have been supported by a table. If the applied force be constant throughout the motion, the work done is measured by the product of the force into the distance through which it acts. The practical unit of work is the *foot-pound*, which is the work done in raising a weight of one pound through a vertical distance of one foot.

It should be noted particularly that the idea of *time* does not enter into work; the work done in raising one ton ten feet high being the same whether a minute or a year be taken to perform it. In the same way, the work done by a cyclist in riding up a

hill of a given height is the same whether he does it slowly or quickly.

The work done in raising a body through a definite height is quite independent of the manner or path of raising, neglecting frictional resistance and considering only the work done against gravity. The work a cyclist does against gravity in ascending a hill of a certain height is quite independent of the *gradient* of the road over which he travels.

Example.—Let the machine and rider weigh 200 lbs., then the work done by the rider in rising 100 feet vertically is 20,000 foot-lbs. If the gradient of the road be known, this can be calculated in another way, which, for the present purpose, is roundabout but instructive. Consider an extreme gradient of one vertical to two on the slope (fig. 58), the length of the hill will be 200 feet. The work done in ascending the hill may be estimated by the product of the force required to push the machine and rider up the hill, into the length of the hill. The machine and rider weigh 200 lbs.; this force acts vertically downwards, and can be resolved into two, one parallel to the road's surface, and one at right angles to it. If $O a$ be set off equal to 200 lbs., and the construction of section 48 be performed, it will be found that the component $b\ O$ required to push the machine and rider up the hill is 100 lbs. The work done will be the product of this force into the distance through which it acts, 200 feet; the result, 20,000 foot-lbs., being the same as before.

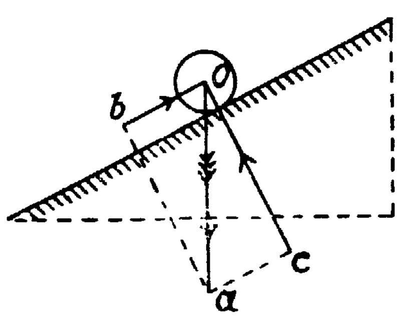


FIG. 58.

This is only the work done against gravity. In riding along a level road there is no work done against gravity, any resistance being made up of the rolling friction of the wheels on the road, air resistance, and the friction of the bearings. These resistances will remain, to all intents and purposes, the same on an incline as on a level. The work done in riding along 200 feet of level road would have to be added to the 20,000 foot-lbs. of work done against gravity, in order to get the *total* work done by the cyclist in ascending the hill.

Generally, the work done by, or against, a force is the product of the force into the projection on the direction of action of the force of the path of the moving body. Thus, if a body move from A to B, and be acted on by the force f, which always retains the same direction, the work done is $f.\ \overline{A\ C}$; $B\ C$ being perpendicular to $A\ C$ (fig. 59).

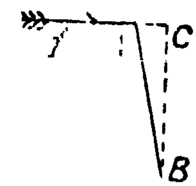


FIG. 59.

The centripetal force acting on a body moving in a circle is always at right angles to the direction of motion; consequently in this case the projection of the path is zero, and no work is done.

In the Simpson lever-chain the pressure of the chain rollers on the teeth of the hub sprocket wheel is at right angles to the surface of the teeth, and consequently makes a considerable angle with the direction of motion of the rollers. In this case, therefore, the projection $A\ C$ (fig. 59), on the line of action of the pressure, of the distance $A\ B$ moved through, is very much less than $A\ B$. The claims of its promoters virtually amount to saying that the work done on the hub by the pull of the chain is $f\ .\ \overline{A\ B}$, whereas the correct value is $f\ .\ \overline{A C}$.

In driving a cycle up-hill, the work done against gravity by the rider at each stroke of the pedal is the product of the total weight and the vertical distance moved through during half a turn of the crank axle. Let the gradient be x parts vertical in 100 on the slope, D the diameter in inches to which the driving-wheel is geared, and W the total weight of machine and rider in lbs. The vertical distance passed through per stroke of pedal is

$$\frac{x}{100} \cdot \frac{\pi D}{2} \text{ inches.}$$

The work done per stroke of pedal is therefore

$$\frac{\pi\, x\, D}{200}\, W \text{ inch-lbs.}$$

$$= \cdot 001309\, x\, D\, W \text{ foot-lbs.} \quad . \quad . \quad . \quad . \quad (2)$$

Table I., on the following page, is calculated from equation (2).

TABLE I.—WORK DONE IN FOOT-LBS. PER STROKE OF PEDAL, IN RAISING 100 LBS. WEIGHT AGAINST GRAVITY.

Diameter, to which driving-wheel is geared	Gradient, parts in 100							
	1	2	3	4	5	6	7	8
Inches								
40	5·24	10·47	15·70	20·94	26·18	31·41	46·65	41·89
45	5·89	11·78	17·67	23·56	29·45	35·34	41·23	47·12
50	6·55	13·09	19·63	26·18	32·72	39·27	47·97	52·36
55	7·20	14·40	21·60	28·79	36·00	43·20	50·39	57·59
60	7·86	15·71	23·56	31·42	39·27	47·12	54·97	62·83
65	8·51	17·02	25·52	34·04	42·54	51·05	59·56	68·08
70	9·16	18·32	27·49	36·65	45·81	54·98	64·14	73·30
75	9·82	19·64	29·45	39·27	49·09	58·90	68·72	78·54
80	10·47	20·94	31·41	41·89	52·36	62·83	73·30	83·78

57. **Power.**—The rate of doing work is called the *power* of an agent, and into its consideration time enters. The standard of power used by engineers is the *horse-power*. Any agent which performs 33,000 foot-lbs. of work in one minute is said to be of 1 H.P. This, Watt's estimate, is in excess of the average power of a horse, but it has been retained as the unit of power for engineering purposes. The average power of a man is about one-tenth that of a horse - that is, equal to 3,300 foot-lbs. per minute.

If V be the speed, in miles per hour, of a cyclist riding up a gradient of x parts in 100, the vertical distance moved through in one minute is

$$\frac{x}{100} \cdot \frac{V \times 5280}{60} = .88\, x\, V \text{ feet,}$$

and the power expended is

$$\cdot 88\, x\, V\, W \text{ foot-lbs. per minute} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Table II. is calculated from equation (3).

58. **Kinetic Energy.**—So far we have dealt with the work done by a force which gives motion to a body against a steady resistance, the speed of the body having no influence on the question, further than it must be the same at the end as at the beginning. If a body free to move be acted on by a force, the work done will be expended in increasing its speed. The work is

TABLE II.—WORK DONE, IN FOOT-LBS. PER MINUTE, IN PUSHING 100 LBS. WEIGHT UP-HILL.

Speed. Miles per hour	Slope, parts in 100							
	1	2	3	4	5	6	7	8
4	352	704	1056	1408	1760	2112	2464	2816
5	440	880	1320	1760	2200	2640	3080	3520
6	528	1056	1584	2112	2640	3168	3696	4224
7	616	1232	1848	2464	3080	3696	4312	4928
8	704	1408	2112	2816	3520	4224	4928	5632
9	792	1584	2376	3168	3960	4752	5544	6336
10	880	1760	2640	3520	4400	5280	6160	7040
11	968	1936	2904	3872	4840	5808	6776	7744
12	1056	2112	3168	4224	5280	6336	7392	8448
13	1144	2288	3432	4576	5720	6864	8008	9152
14	1232	2464	3696	4928	6160	7392	8624	9856
15	1320	2640	3960	5280	6600	7920	9240	10560
16	1408	2816	4224	5632	7040	8448	9856	11264
17	1496	2992	4488	5984	7480	8976	10472	11968
18	1584	3168	4752	6336	7920	9504	11088	12672
19	1672	3344	5016	6688	8360	10032	11704	13376
20	1760	3520	5280	7040	8800	10560	12320	14080

stored in the moving body, and can be restored in bringing the body again to rest. This stored work is called *kinetic energy*.

59. **Potential Energy.**—Newton's first law of motion expresses the idea of permanence of motion of a body unless altered by applied forces. If the speed of a body on which no force acts remains constant, its kinetic energy must also remain constant. If a body free to move is acted on by a force, the work done by the force is stored up as kinetic energy. If work is done by moving the body against the resistance of a force which is constant in magnitude and direction, whatever be the direction of motion, the work is expended in changing the position of the body. For example, in raising a body from the ground, the resistance overcome is its weight, which always acts vertically downwards, whether the body be at rest or moving upwards or downwards. If the body be lowered by suitable means to the ground, the work done in raising it is again restored. The body

at rest a certain height above the ground possesses therefore an amount of energy due to its position; this is called *potential energy*. If the body be allowed to fall freely under the action of gravity, at the instant of reaching the ground it possesses no potential energy, but kinetic energy due to its speed. Its initial store of potential energy has been all converted into kinetic energy.

60. **Conservation of Energy.**—The great principle of conservation of energy is an assertion that energy cannot be created or destroyed. This is one of the most comprehensive generalisations that has been deduced from our observations of natural phenomena. Applied to the case of a body moving under the action of force without any frictional resistance, it asserts that the sum of the kinetic and potential energies is constant. A cyclist riding down a short hill with his feet off the pedals and not using the brake, will have a greater speed at the bottom than at the top, part of the potential energy due to the high position at the top of the hill being converted into kinetic energy at the bottom. If another short hill of equal height has to be ascended immediately, the kinetic energy at the bottom gets partially converted into potential energy at the top; the rider arriving at the top of the second hill with the same speed as he left the first. The friction of the air, tyres, and bearings has been neglected in the above discussion. If the rider just work hard enough to overcome these resistances as on a level road, the above statement will be strictly true.

Applied to mechanism used to transmit and modify power, the principle of the conservation of energy is sometimes quoted, 'No more work can be got out at one end of a machine than is put in at the other.' The work got out will be exactly equal to that put into the machine, provided the friction of the machine is zero, an ideal state of things sometimes closely approached, but never actually attained in practice. The chronic inventor of cycle driving-gears might save himself a great deal of trouble by mastering this principle.

61. **Frictional Resistance.**—It is a matter of every-day experience that a moving body left to itself will ultimately come to rest, thus apparently contradicting Newton's first law. A flat stone moved along the ground comes to rest very soon. If the

stone be round, it may roll along the ground a little longer, while a bicycle wheel with pneumatic tyre set off with the same speed will continue its motion for a still longer period. A wheel set rapidly rotating on its axis will gradually come to rest. If the wheel be supported on ball-bearings, the motion may continue for a considerable fraction of an hour, but ultimately the wheel will come to rest. In all these cases there is a force in action opposing the motion, the force of *friction*, which is always called into play when two bodies move in contact with each other. The amount of friction depends on the nature of the surfaces in contact. The friction is very great with the flat stone sliding along the ground, is less with the rolling stone, and still less with the pneumatic-tyred wheel. The friction of a ball-bearing may be reduced to a very small amount, but cannot be entirely abolished; the less the friction, the longer the motion persists. The air also offers a considerable resistance to the motion, which varies with the speed. If a wheel with ball-bearings could be set in rapid rotation under a large bell-jar from which the air had been exhausted by an air-pump, the motion of the wheel might persist for several hours, and thus give a close approximation to an experimental verification of Newton's first law of motion. The movement of the planets through space affords the best illustration of the permanence of motion.

62. **Heat.**—The force of friction is thus seen to diminish the kinetic energy of a moving body, while if the body move in a horizontal plane, its potential energy remains the same throughout, and energy is said to be *dissipated*. The energy dissipated is not destroyed, but is converted into *heat*, the temperature of the bodies in contact being raised by friction. Heat is a form of energy, and the conversion of mechanical work by friction into heat is a matter of every-day experience; conversely, heat can be converted into mechanical work. Steam engines, gas-engines, and oil-engines are machines in which this conversion is effected. Heat due to friction is energy in a form which cannot be utilised in the machine in which it arises; hence popularly engineers speak of the work *lost* in friction, such energy being in a useless form.

In riding down-hill the potential energy of the machine and

rider gets less ; if the speed remains the same, the kinetic energy remains the same, and the potential energy is dissipated in the form of heat. If a brake be used, the heat appears at the brake-block and the wheel on which it rubs. If back-pedalling be employed, the same amount of heat is expended in heating the muscles of the legs, though the other physiological actions going on may be such as to render the detection or measurement of this heat difficult.

Mechanical Equivalent of Heat.—The conversion of heat into work, and work into heat, takes place at a certain definite rate. 780 foot-pounds of work are equivalent to one unit of heat ; the unit of heat being the quantity of heat required to raise the temperature of one pound of water one degree Fahrenheit. Thus, in descending a hill 100 feet high, a rider and machine weighing 200 lbs. would convert 20,000 foot-lbs. of work into $\frac{20000}{780} = 25{\cdot}6$ units of heat. If this could all be collected at the brake-block, it would be sufficient to raise the temperature of one pound of water 25·6 degrees.

CHAPTER VIII

DYNAMICS (*continued*).

63. **Dynamics of a Particle.**—A particle, an ideal conception in the Science of Mechanics, is a heavy body of such small dimensions that it may be considered a point. If a particle of mass m initially at rest, but free to move, be acted on for time t by a constant force f, we have seen (sec. 16) that the speed v imparted is such that

$$f\,t = m\,v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

or

$$f = \frac{m\,v}{t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$f = m\,a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

a being the acceleration, or rate of change of speed, $m\,v$ is the momentum acquired in time t, hence $\frac{m\,v}{t}$ is the momentum acquired in unit of time, and (1) is equivalent to defining force as 'rate of change of momentum.'

Let s be the distance traversed in the time t; then since the *average* speed is half the speed at the end of the period,

$$s = \tfrac{1}{2}\,v\,t = \tfrac{1}{2}\,a\,t^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The work done during the period is $f\,s$, and

$$f\,s = \tfrac{1}{2}\,v\,f\,t = \tfrac{1}{2}\,m\,v^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the particle has initially a speed v_0, equations (1), (3) and (4) become

$$f\,t = m\,(v - v_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$s = \tfrac{1}{2}\,(v + v_0)\,t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$f\,s = \tfrac{1}{2}\,m\,(v^2 - v^2_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Kinetic Energy.—The work done by the force has been expended in giving the body its speed v, and the body in coming to rest can restore exactly the same amount of work. The product $\frac{1}{2} m v^2$ is called the *kinetic* energy of the moving body ; it may be denoted by the symbol E.

The units employed above are all absolute units. The unit of kinetic energy in (4) is the foot-poundal ; in foot-pounds the kinetic energy is

$$E = \frac{m v^2}{2 g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Falling Bodies.—A body falling freely under the action of gravity is a special case of the above. Let the mass m be one pound, the force acting on the body is 1 lb. weight, *i.e.* g poundals. Writing g instead of f, and $m = 1$, in equations (1)-(4) the formulæ for falling bodies are obtained.

64. **Circular Motion of a Particle.**—Let the particle be constrained to move in a circle of radius r, and be acted on by a force of constant magnitude f, which is always in the direction of the tangent to the path of the particle ; then since the radial force does no work, equations (1) to (7) still hold. Multiply both sides of (1) by r, then

$$f r = \frac{m v r}{t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$f r$ is the moment of the applied force about the axis of rotation, $m v$ is the momentum, $m v r$ the *moment of momentum* or *angular momentum* ; hence the moment of a force is equal to the rate of change of angular momentum it produces.

If ω be the angular speed and θ the angular acceleration of the particle about the axis at the end of the time t, $v = \omega r$, $\theta = \frac{\omega}{t}$, and (9) may be written

$$f r = \frac{m \omega r^2}{t} = m r^2 \theta \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The product $m r^2$ is the *moment of inertia* of the particle about the axis of rotation, and may be denoted by i ; (10) may then be written

$$f r = i \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

That is, the moment of the force is equal to the product of the moment of inertia of the body on which it acts and the angular acceleration it produces.

Equation (4) becomes, for this case,

$$e = f\,s = \tfrac{1}{2}\,m\,v^2 = \tfrac{1}{2}\,m\,r^2\,\omega^2 = \tfrac{1}{2}\,i\,\omega^2 \quad . \quad . \quad . \quad (11)$$

That is, the kinetic energy of a particle moving in a circle is half the product of its moment of inertia about the centre and the square of its angular speed.

(9) may be written

$$f\,t\,r = m\,v\,r = m\,r^2\,\omega = i\,\omega \quad . \quad . \quad . \quad . \quad (12)$$

$f\,t$ is the impulse of the force ; therefore the moment of the impulse is equal to the product of the moment of inertia of the particle and the angular speed produced by the impulse.

65. **Rotation of a Lamina about a Fixed Axis Perpendicular to its Plane.**—A rigid body of homogeneous material may be considered to be made up of a great number of particles, all of equal mass uniformly distributed. A rigid lamina is a rigid body of uniform, but indefinitely small, thickness lying between two parallel planes ; a flat sheet of thin paper is a physical approximation to a lamina. Let O (fig. 60) be the fixed axis of rotation, perpendicular to the plane of the paper ; let A be any particle of the lamina distant r from O. Then using the same notation, equations (9) to (12) hold for the particle A, the acting force f being always at right angles to the radius $O\,A$. Now the rigid lamina may be considered made up of a number of heavy particles like A, embedded in a rigid weightless frame. Instead of the force f acting directly at A, suppose a force p act at a point B of the frame in a direction at right angles to $O\,B$. Let $O\,B = l$, then if

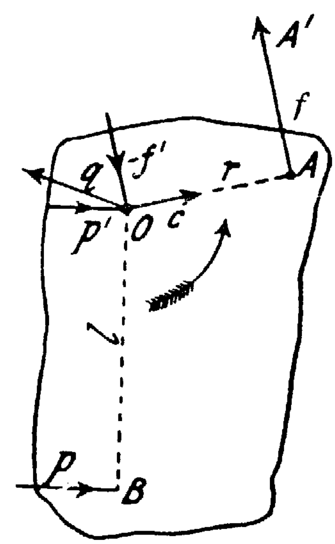


FIG. 60.

$$p\,l = f\,r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

the effects of the forces f and p in turning the weightless frame and heavy particle A about the centre O are exactly the same ; the motion of A is unaltered by the substitution.

Also, if d be the space passed over during the period by the point B,

$$\frac{d}{l} = \frac{s}{r} \text{ or } s = \frac{d\,r}{l}, \text{ and}$$

therefore

$$f\,s = \frac{p\,l}{r} \cdot \frac{d\,r}{l} = p\,d.$$

Substituting in (10), (11) and (12) they may be written

$$p\,l = i\,\theta \quad . \quad . \quad . \quad . \quad (14)$$
$$e = p\,d = \tfrac{1}{2}\,i\,\omega^2 \quad . \quad . \quad . \quad . \quad (15)$$
$$p\,l\,t = i\,\omega \quad . \quad . \quad . \quad . \quad (16)$$

Let i_1, i_2 . . . . be the moments of inertia of the heavy particles A_1, A_2 . . . . of which the lamina is composed: p_1, p_2, ... the corresponding forces at the point B required to give them their actual motions; then for all the particles, (14), (15) and (16), may be written

$$(p_1 + p_2 + \quad . \quad . \quad . \quad)\,l = (i_1 + i_2 + \quad . \quad . \quad . \quad)\,\theta$$
$$(p_1 + p_2 + \quad . \quad . \quad)\,d = \tfrac{1}{2}\,(i_1 + i_2 \quad . \quad . \quad . \quad)\,\omega^2$$
$$(p_1 + p_2 + \quad . \quad . \quad . \quad)\,l\,t = (i_1 + i_2 + \quad . \quad . \quad . \quad)\,\omega$$

l, t, θ and ω being the same for all the particles. Let $I = (i_1 + i_2 + \quad . \quad . \quad . \quad)$, then I is the moment of inertia of the lamina about the axis O; let $(p_1 + p_2 + \quad . \quad . \quad . \quad) = P$, then P is the actual force applied at the point B of the lamina; let $(e_1 + e_2 + \quad . \quad . \quad . \quad) = E$, then E is the kinetic energy of the lamina; and the above equations may be written

$$P\,l = I\,\theta \quad . \quad . \quad . \quad . \quad (17)$$
$$E = \tfrac{1}{2}\,I\,\omega^2 \quad . \quad . \quad . \quad . \quad (18)$$
$$P\,l\,t = I\,\omega \quad . \quad . \quad . \quad . \quad (19)$$

$P\,l$ is the magnitude of the applied turning couple.

66. **Pressure on the Fixed Axis.**—In the above investigation the pressure on the axis at O has been neglected, since whatever be its value, its moment about O is zero, and it does not, therefore, influence the speed of rotation. It is, however, desirable to know the pressure on the bearings of the rotating body; we therefore proceed to investigate it. Consider only the particle A, connected by the rigid weightless frame to B and O; if the force p at B

gives A its tangential acceleration, the weightless frame must press on the particle A with a force f, in the direction at right-angles to r, and the particle A must react on the frame with an equal and opposite force $-f$. But the particle A also presses on the frame with the centrifugal force $c = \frac{m\,v^2}{r} = m\,\omega^2\,r$, in the direction of the radius r. The frame being weightless must be in equilibrium under the forces acting on it; since, by (2), a finite force, however small, acting on a body of zero mass would produce infinite acceleration. These forces are: $-f$ at A, p at B, the reaction q of the axis at O, and the centrifugal force c, which also acts through O. But the forces $-f$ at A, and p at B, are equivalent to equal and parallel forces at O, and the couples $-fr$ and pl. The couples equilibrate each other, therefore the four forces $-f^1$, p^1, q and c at O are in equilibrium. Therefore,

$$\text{vector } q = \text{vector } f - \text{vector } p - \text{vector } c \quad . \quad . \quad . \quad (20)$$

Let Q be the resultant reaction of the fixed axis on the lamina, due to the particles A_1, A_2, . . . of which it is composed, *i.e.*—

$$\text{vector } Q = \text{sum of vectors } q_1, q_2 \quad . \quad . \quad .$$

Similarly, let

$$\text{vector } P = \text{sum of vectors } p_1, p_2 \quad . \quad . \quad .$$
$$\text{vector } F = \text{sum of vectors } f_1, f_2 \quad . \quad . \quad .$$
$$\text{vector } C = \text{sum of vectors } c_1, c_2, c_3 \quad . \quad .$$

Then, adding equations (20) for all the particles A_1, A_2 . . .,

$$\text{vector } Q = \text{vector } F - \text{vector } P - \text{vector } C \quad . \quad . \quad (21)$$

But by (10)—

$$\text{vector } F = m\,\theta \times \text{vector sum } (r_1 + r_2 + \; . \; . \; . \; .)$$

And the vector sum $(r_1 + r_2 + \; . \; . \; . \; r)$ is the vector $n \,.\, OG$; G being the mass-centre of the lamina (fig. 61), and n the number of particles, each of mass m, it contains.

Therefore,

$$\text{vector } F = M\theta \,.\, O\overline{G} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

M being the total mass of the lamina. The component forces f_1, f_2 . . . acting at right-angles to the corresponding vectors r_1,

r_2 . . ., the resultant force F will act at right angles to the resultant vector $O\,G$. Similarly,

$$\text{vector } C = M\,\omega^2 \,.\, \overline{O\,G} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

the force C acting along $O\,G$.

Now, from (13) and (10) $p = \dfrac{f\,r}{l} = \dfrac{m\,r^2\,\theta}{l}$.

The vectors p are all in the same direction, at right angles to $O\,B$, and are therefore added like scalars. Therefore,

$$\text{vector } P = \frac{\theta}{l} \times \text{sum } (m_1\,r_1{}^2 + m_2\,r_2{}^2\ .\ .\ .) = \frac{I\,\theta}{l} \quad . \quad (24)$$

Substituting these values in (21), the reaction Q (fig. 61) of the fixed axis is the resultant of:—A force at O equal and parallel to that required to accelerate the mass M supposed concentrated at G; a force at O equal, opposite and parallel to the applied force P; the centripetal force $M\,.\,\omega^2\,.\,\overline{O\,G}$, acting along $G\,O$.

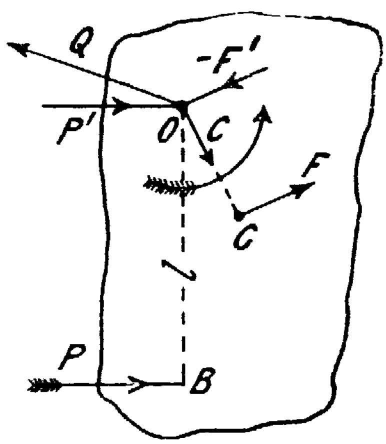


FIG. 61.

From (21) many important results can be deduced. Let a couple act on a rigid lamina quite free to move in its plane; then $P = 0$, $Q = 0$; and (21) becomes

$$\text{vector } F - \text{vector } C = 0.$$

But the vectors F and $-\,C$ are at right angles; their sum can only be zero when each is zero. This is the case when $\overline{O\,G} = 0$ —see (22) and (23)—that is, when the mass-centre and the axis of rotation coincide. Hence a couple applied to a lamina free to move causes rotation about its mass-centre.

67. **Dynamics of a Rigid Body.**—Equations (17), (18) and (19) are applicable to the rotation of any rigid body about a fixed axis. Equations (21) to (24) are applicable if the rigid body is symmetrical about a plane perpendicular to the axis of rotation; this includes most cases occurring in practical engineering. But in a non-symmetrical body, *e.g.* a pair of bicycle cranks and their axle, the resultant pressure on the bearings cannot be expressed

as a single force, but is a couple. Thus, such a rigid body, if perfectly free, will turn about an axis, in general, not parallel to that of the acting couple.

From (23), the centrifugal pressure on the fixed axis of any rigid body is the same as if the whole mass were concentrated at the mass-centre G. If the mass-centre lies on the axis of rotation, the centrifugal pressure is zero. Hence the necessity of accurately balancing rapidly revolving wheels. In this case also (21) becomes $Q = -P$, *i.e.* the pressure on the bearing is equal and parallel to the applied force, provided Q can be expressed as a single force. If only a couple be applied, $P = 0$, and the pressure on the bearings is zero. In a rapidly rotating wheel with horizontal axis, P is the weight of the wheel; with vertical axis $P = 0$, the weight acting parallel to the axis.

The motion of a rigid body can be expressed (sec. 41) as a translation of its mass-centre, and a rotation about an axis passing through its mass-centre. Any applied force is equivalent to an equal parallel force at the mass-centre and a couple of transference. The rotation about the mass-centre is the effect of this couple. Hence, the turning effect of any system of forces acting on a free rigid body is the same as if its mass-centre were fixed. Since the resultant couple does not influence the motion of the mass-centre, the motion of the mass-centre of a rigid body under the action of any system of forces is the same as if equal parallel forces were applied at the mass-centre.

The kinetic energy of any moving body is the sum of the energy due to the speed of its mass-centre, and the energy due to its rotation about the mass-centre.

Moments of Inertia.—If M be the total mass of a rigid body, its moment of inertia may be expressed $I = Mk^2$; and k is called the radius of gyration. The I about an axis through the mass-centre is least: let it be denoted by I_0; that about any parallel axis distant h is

$$I = I_0 + Mh^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The values of I for a few forms may be given here. For a thin ring of radius r and mass M rotating about its geometric axis, $I_0 = Mr^2$. This is approximately the case of the rim and tyre of

a bicycle wheel. For the same ring rotating about an axis at its circumference, as in rolling along the ground, $I = 2\, M r^2$.

For a bar of length l rotating about an axis through its end perpendicular to its own axis, $I = \dfrac{M l^2}{3}$. This is approximately the case of the spokes of a bicycle wheel.

For a circular disc of uniform thickness and radius r rotating about its geometric axis, $I_0 = \dfrac{M r^2}{2}$. For the same disc rolling along the ground, $I = \dfrac{3}{2} M r^2$.

68. **Starting in a Cycle Race.**—The work done by a rider at the beginning of a race is nearly all expended in giving himself and machine kinetic energy, the frictional resistances being small until a high speed is attained. If the winning-post be passed at top speed, the kinetic energy is practically not utilised. In a short distance race, this kinetic energy may be large in comparison to the energy employed in overcoming frictional resistances. The kinetic energy of translation of the machine and rider is $\dfrac{W v^2}{g\ r}$ foot-lbs., W being the total weight. Hence, a light machine, other things being equal, is better than a heavy one for short races. Further, there is the kinetic energy of rotation of the wheels and cranks. For the rims and tyres this is nearly equal to their translational kinetic energy; therefore, at starting a race, one pound in the rim and tyres is equivalent to two pounds in the frame. In comparing racing machines for sprinting, the weight of the frame, added to twice that of the rims and tyres, would give a better standard than the weight of the complete machine. The pneumatic tyre, with its necessarily heavier rim, is, in this respect, inferior to the old narrow solid tyre. Of course, once the top speed is attained, the weight of the parts has no direct influence, but only so far as it affects frictional resistances.

69. **Impact and Collision.**—If two bodies moving in opposite directions collide, their directions of motions are apparently changed instantaneously; but, as a matter of fact, the time during which the bodies are in contact, though extremely short, is still appreciable. The magnitude of the force required to generate

velocity in a body, or to destroy velocity already existing, is inversely proportional to the time of action; if the time of action be very short, the acting force will be very large. Such forces are called *impulsive forces*.

Now in the case of colliding bodies, such as a pair of billiard balls, it is impossible either to measure f or t; but the mass m of one of the balls, and its velocities v_0 and v before and after collision, may easily be measured. The expression on the right-hand side of (5) denotes the increase of momentum of the body due to the collision; the product $f\,t$ on the left-hand side is called the *impulse*; therefore, from (5), the impulse is equal to the change of momentum it produces.

We shall now have to examine more minutely the nature of the forces between two bodies in collision: At the instant that the bodies first come into contact they are approaching each other with a certain velocity. Suppose A (fig. 62) to be moving to the right, and B to the left; immediately they touch, the equal impulsive forces f_1 and f_2 will be called into action, and will oppose the motions of A and B respectively. The parts of the bodies in the neighbourhood of the place of contact will be flattened, and this flattening will increase until the relative velocity of the bodies is zero. The time over which this action extends is called the *period of compression*. If the bodies are elastic, they will tend to recover their original shapes, and will therefore still press against each other; the forces now tending to give the bodies a relative velocity in the direction opposite to their original relative velocity. These impulsive forces will be in action until the original shape has been recovered and the bodies leave each other. The time over which this action extends is called the *period of restitution*; and the total impulse may be conveniently divided into two parts, the impulse of compression and the impulse of restitution.

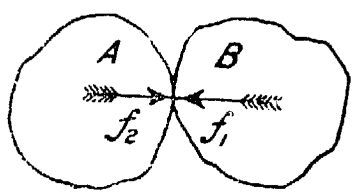


FIG. 62.

Index of Elasticity.—Now it is an experimental fact that in bodies of given material the impulse of restitution bears a constant ratio to the impulse of compression; this ratio is called the *index of elasticity*. A perfectly elastic material has its index of elasticity unity; in an inelastic body the index of elasticity is zero; if the

index of elasticity lies between zero and unity, the body is imperfectly elastic. The index of elasticity e is, for balls of glass $\frac{15}{16}$, for balls of ivory $\frac{8}{9}$, and for balls of steel $\frac{5}{9}$. These are the values given by Newton, to whom the theory of collision of bodies is due.

Conservation of Momentum.—In figure 62, the force f_1 at any instant acting on A is exactly equal to the force f_2 acting on B; the total impulse on A is therefore equal to the total impulse on B; and as they are in opposite directions their sum is zero. Thus, the momentum of the system is the same after collision as before it. This is true whether the bodies are inelastic, imperfectly elastic, or perfectly elastic. If two bodies of mass m_1 and m_2, moving with velocities v_0' and v_0'' respectively, collide, their velocities after collision can be easily determined, if the index of elasticity e is given. For cyclists, the most important case is when one of the bodies is rigidly fixed; in other words, when m_2 is infinite and v_0'' zero. Let, as before, the mass of the finite body be m, its velocities before and after collision with the infinite body be v_0 and v; then before collision its momentum is $m\, v_0$. Let C be the impulse of compression; then since at the end of the compression period the velocity is zero, we get by substitution in (1)

$$C = m\, v_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The impulse of restitution, by definition, is $e\, C$; therefore, if v be the velocity of the body after collision, we have

$$e\, C = -\, m\, v_0$$

Substituting the value of C from (26), we get

$$v = -\, e\, v_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

That is, the speed of rebound is equal to the speed of impact multiplied by the index of elasticity. The speed of rebound is therefore always less than the speed of impact.

This result at first sight seems to be contradictory to the principle of the conservation of momentum, but remembering that the mass of the fixed body may be considered infinite, and its velocity zero, its momentum is

$$\infty \times 0,$$

an expression which may represent *any* finite magnitude. We may say the fixed body gains the momentum lost by the moving

body by the collision. For example, when a ball falls vertically and rebounds from the ground, the earth as a whole is displaced by the collision.

Loss of Energy.—The kinetic energy of the moving body before impact is

$$\frac{m v_0^2}{2 g} \text{ foot-lbs.};$$

the kinetic energy after impact is

$$E = \frac{e^2 m v_0^2}{2 g} = e^2 E_0 \quad . \quad . \quad . \quad . \quad . \quad (28)$$

The loss of energy due to collision is thus

$$(1 - e^2) E_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

70. **Gyroscope.**—Let a wheel W (fig. 63), of moment of inertia I, be set in rapid rotation on a spindle S, which can be balanced by means of a counterweight w, on a pivot support T (fig. 63). If a couple C, formed by two equal and opposite vertical forces F_1 and F_2 acting at a distance l, be applied to the spindle, tending to make it turn about a horizontal axis, it is found that the axis of the spindle turns slowly in a horizontal plane. This motion is called 'precession.' This phenomenon, which,

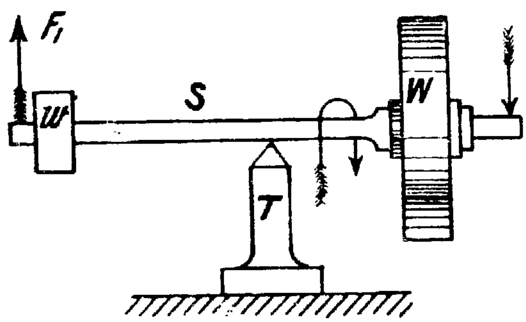


FIG. 63.

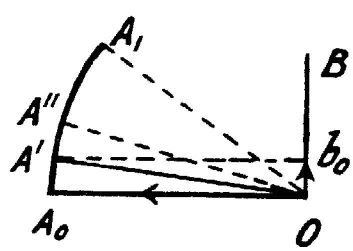


FIG. 64.

when observed for the first time, appears startling and paradoxical, can be strikingly exhibited by removing the counterweight w, so that statically the spindle is not balanced over its support. The explanation depends on the composition of rotations. Figure 64 is a plan showing the initial direction $O A_0$ of the axis of rotation of the wheel W. The initial angular momentum of the wheel can be represented to any convenient scale by the length $O A_0$. The couple C tends to give the wheel a rotation about the axis

OB at right angles to OA_0. If this couple C acts for a very short period of time, t_1, the angular momentum it produces about the axis OB is Ct_1. This may be represented to scale by Ob_0. The resultant angular momentum of the wheel at the end of the time, t_1, may therefore be represented in magnitude and direction by OA^1. If the time t_1 be taken very small, OA^1 is practically equal to OA_0, and the only effect of the couple C is to alter the *direction* of the axis of rotation. At the end of a second short interval of time, t_2, it may be shown in the same manner that the axis of rotation is OA'', $A'A''$ being at right angles to OA'. At the end of one second the increment of the angular momentum is numerically equal to C, and may be represented by the arc A_0A_1; thus at the end of one second the axis of rotation is OA_1. Let θ be the angular speed of precession, then θ is numerically equal to the angle A_0OA_1, *i.e.*,

$$\theta = \frac{\text{arc } A_0A_1}{\text{radius } OA_0} = \frac{C}{I\omega} \quad \text{. . . . . . (30)}$$

or

$$C = I\omega\theta = Mk^2\omega\theta$$
$$= Mvk\theta \quad \text{. . . . . (31)}$$

where M is the mass and k the radius of gyration of the wheel, and v the linear speed of a point on the wheel at radius k.

In drawing the diagram (fig. 64) care should be taken that the quantities OA_0 and Ob_0 are marked off in the proper direction. If the rotation of the wheel when viewed in the direction OA_0 appear clock-wise, it may be considered positive; similarly, the rotation which the couple C tends to produce, appears clock-wise when measured in the direction Ob_0, and is therefore also considered positive. If the couple C were of the opposite sign, the increment of angular momentum Ob_0 would be set off in the opposite direction, and the precession would also be in the opposite direction.

The geometrical explanation of this phenomenon is almost the same as that given for centrifugal force in the case of uniform motion in a circle.

A cyclist can easily make an experiment on precession without any special apparatus as follows: Detach the front wheel from a

bicycle, and, supporting the ends of the hub spindle between the thumb and first fingers of each hand, set it in rotation by striking the spokes with the second and third fingers of one hand. On withdrawing one hand the wheel will not fall to the ground, as it would do if at rest, but will slowly turn round, its axis moving in a horizontal plane. As the speed of rotation gradually gets less owing to friction of the air and bearings, the speed of precession gets greater, until the wheel begins to wobble and ultimately falls.

71. **Dynamics of any System of Bodies.**—The forces acting on any given system of bodies may be conveniently divided into 'external' and 'internal'; the former due to the action of bodies external to the given system, the latter made up of the mutual actions between the various pairs of bodies in the given system. The latter forces are in equilibrium among themselves; that is, the force which any body A exerts on any other body B of the system is equal and opposite to the force exerted by B on A. The motion of the mass-centre of the given system is therefore unaffected by the internal forces, and some of the results of section 67 can be extended to any system of bodies, thus:

The motion of the mass-centre of a system of bodies under the action of any system of forces is the same as if equal parallel forces were applied at the mass-centre.

The turning effect of a system of forces acting on any system of bodies is the same as if the mass-centre of the system were fixed.

The kinetic energy of any system of bodies is the sum of the kinetic energies due to: (*a*) the total mass collected at, and moving with the same speed as, the mass-centre of the system; (*b*) the masses of the various bodies concentrated at their respective mass-centres, and moving round the mass-centre of the system; (*c*) the rotations of the various bodies about their respective mass-centres.

Example.—If a retarding force be applied to the side wheel of a tricycle, the diminution of speed is the same as if the force were applied at the mass-centre of the machine and rider, while the turning effect on the system is the same as if the machine were at rest. (See chap. xviii.)

CHAPTER IX

FRICTION

72. **Smooth and Rough Bodies.**—If two perfectly *smooth* bodies are in contact, the mutual pressure is always in a direction at right angles to the surface of contact. Thus a smooth stone resting on the smooth frozen surface of a pond presses the ice vertically downwards, and the reaction from the ice is vertically upwards. If a horizontal force be applied to the stone it will move horizontally, the mutual pressure between it and the ice offering little resistance to this motion. A smooth surface may be defined as one which offers no resistance to the motion of a body upon it. No *perfectly* smooth surface exists in nature, but all are more or less rough, and offer resistance to the motion of a body upon them. This resistance is called *friction*.

Friction always acts in the direction opposed to the motion of a body, and thus tends to bring it to rest. In all machinery, therefore, great efforts are made to reduce the friction of the moving parts to the least possible value. In bearings of machinery friction is a most undesirable thing, but in other cases it may be a most useful agent. Without friction, no nut would remain tight after being screwed up on its bolt ; railways would be impossible ; and in cycling, not only would it be impossible to ride a bicycle upright on account of side-slip, but not even a tricycle could be driven by its rider along the ground, as the driving-wheels would simply skid.

73. **Friction of Rest.**—The greatest possible friction between two bodies is measured by the force parallel to the surface of contact which is just necessary to produce sliding. If a force acting parallel to the surface be less than this amount, the bodies will remain at rest.

It is found by experiment that friction varies with the nature of the surfaces of contact; is proportional to the mutual normal pressure, and is independent of the area of the surface of contact so long as the pressure remains the same. When sliding motion actually takes place, the friction is often less than when the bodies are at rest in a state just bordering on motion.

74. **Coefficient of Friction.**—Let P be the force perpendicular to the surface of contact with which two bodies are pressed together, and F the force parallel to the surface which is just necessary to make one slide on the other. Then, as stated above, it is found experimentally that F is proportional to P. The ratio of F to P is called the *coefficient of friction* for the particular surfaces in contact; this is usually denoted by the Greek letter μ. The coefficient of friction for iron on stone varies from ·3 to ·7; for wood on wood from ·3 to ·5; for metal on metal from ·15 to ·25; while for india-rubber on paper the author has observed values greater than 1·0.

Angle of Friction.—If two bodies be pressed together with a force P, making an angle θ with the normal to the surface, its components P_1, perpendicular to, and P_2, parallel to, the surface can be readily obtained by drawing. If $\frac{P_2}{P_1}$ be less than μ, no sliding will take place, but if $\frac{P_2}{P_1}$ be greater than μ, sliding will occur. The angle θ at which sliding just occurs is called the *angle of friction.*

If one of the bodies be an inclined plane and the other a body of weight W resting on it, the force P pressing them together is vertical, and therefore inclined at an angle θ to the normal to the surface; the angle θ of the inclined plane at which the body will first slide down is evidently the same as the angle of friction, and is sometimes called the *angle of repose.*

75. **Journal Friction.**—It has been established by experiment that the friction of two bodies sliding on each other at moderate speeds, under moderate pressures, and with the surfaces either dry or very slightly lubricated, is independent of the speed of sliding and of the area of the surfaces of contact, and is simply

proportional to the mutual pressure. The experiments on which the laws of friction rest were made by Morin in 1831. With well-lubricated surfaces, such as in the bearings of machinery, the laws of friction approximate to those relating to the friction of fluids. Mr. Tower made experiments, for the Institution of Mechanical Engineers, on the friction of cylindrical journals, which showed that when the lubrication of the bearing was perfect, the total friction remained constant for all loads within certain limits. The coefficient of friction is therefore inversely proportional to the load. The total friction also varies directly as the square root of the speed. The coefficient of friction may therefore be represented by a formula

$$\mu = C\frac{\sqrt{v}}{P} \quad \ldots \ldots \ldots \quad (1)$$

These experiments clearly show that with perfect lubrication the journal does not actually touch the bearing, but floats on a thin film of oil held between the two surfaces. The most perfect form of lubrication is that in which the journal dips into a bath of oil. The ascending surface drags with it a supply of oil, and so the film between the journal and its bearing is constantly renewed. If the lubrication is imperfect the coefficient of friction rises considerably, the conditions approaching then those which hold with regard to solids.

The journal experimented on was 4 in. diameter by 6 in. long. With oil-bath lubrication, running at 200 revolutions per minute, and with a total load on the journal of 12,500 lbs., the total friction at the surface of the journal was 12·5 lbs., giving a coefficient of friction of ·0010. With a total load of 2,400 lbs. the total friction at the surface of the journal was 13·2 lbs., giving a coefficient of friction of ·0055.

76. **Collar Friction.**—The research committee of the Institution of Mechanical Engineers also carried out some experiments on the friction of a collar bearing. The collar was a ring of mild steel, 12 in. inside and 14 in. outside diameter, and bore against gun-metal surfaces. The pressure per square inch which such a bearing could safely carry was far less than in a cylindrical journal; the lowest coefficient of friction was ·031, corresponding to a

pressure of 90 lbs. per square inch, and a speed of 50 revolutions per minute. μ was practically constant, its average value being about ·036.

The much higher coefficient of friction in a collar than in a cylindrical bearing is no doubt due to the fact that a thin film of oil cannot be held between the surfaces, and be continually renewed.

77. **Pivot Friction.**—The relative motion of the surfaces of contact in a pivot bearing is one of rotation about an axis at right angles to the common surface of contact. Let figure 65 represent plan and elevation of a pivot bearing, o being the axis of rotation and ω the angular speed. The linear speed of rubbing of any point at a radius r from the centre will be ωr. Let W be the total load on the pivot, D its diameter, and R its radius. If we assume the pressure to be uniformly distributed over the surface of contact, the pressure per square inch will be,

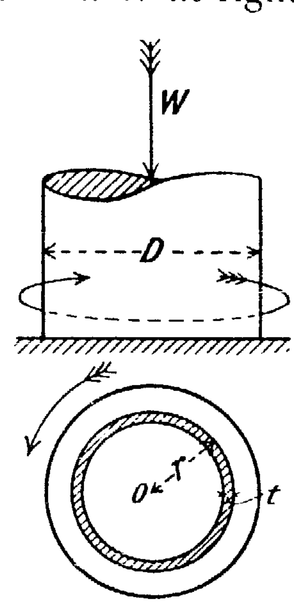


FIG. 65.

$$p = \frac{4}{\pi}\frac{W}{D^2} \quad . \quad . \quad . \quad .$$

The area of a ring of mean radius r and width t is $2\,\pi\,r\,t$. The frictional resistance due to the pressure on this ring is $2\,\mu\,\pi\,r\,t\,p$, and the moment about the centre O is $2\,\mu\,\pi\,r^2 t\,p$. Summing the moments for all the rings into which the bearing surface of the pivot may be divided, the moment of the frictional resistance of the pivot is

$$\frac{2\,\mu\,\pi\,R^3\,p}{3} = \frac{\mu\,W\,D}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

That is, the frictional resistance due to the load W may be supposed to act at a distance from the centre of one-third the diameter of the pivot.

If the diameter be very small, the average linear speed of rubbing, and therefore also the total work lost in friction, will be small. The work lost in friction is converted into heat, and the heat must be carried away as fast as it is generated, or the temperature of the bearing will rise and the surface will seize.

The pressure per square inch a bearing may safely carry will thus depend on the quantity of heat generated per unit of surface, and therefore on the speed of rubbing. This speed being small in pivot bearings, they may safely work under greater pressure than collar bearings.

It will be shown (chapter xxv.) that the motion of a ball in a ball bearing is compounded of rolling and spinning. Rolling friction is discussed in section 78.

Spinning friction of a ball on its path is analogous to pivot friction, with the exception that the surfaces have contact only at a point when no load is applied. When the ball is pressed on its path by a force W (fig. 66) the surfaces in the immediate neighbourhood of the geometrical point of contact o are deformed, and contact takes place over an area $a\ o\ b$. The intensity of pressure is probably greatest at o, and diminishes to zero at a and b. The frictional resistance thus ultimately depends on the diameter of the ball, its hardness, the radius of curvature of its path, the load W as well as the coefficient of friction. No experiments on the spinning friction of balls have been made, to the author's knowledge, though they would be of great use in arriving at a true theory of ball-bearings.

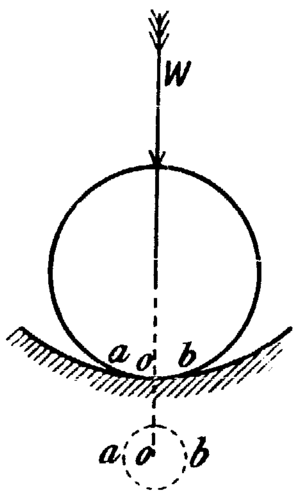


FIG. 66.

78. **Rolling Friction.**—When a cylindrical roller rolls on a perfectly horizontal surface there is a resistance to its motion, called *rolling friction*. Professor Osborne Reynolds has investigated the nature of rolling resistance, and he finds that it is due to actual sliding of the surfaces in contact. No material in nature is absolutely rigid, so that the roller will have an *area* of contact with the surface on which it rolls, the extent of which will vary with the material and with the curvature of the surfaces in contact. Figure 67 shows what takes place when an iron roller rests on

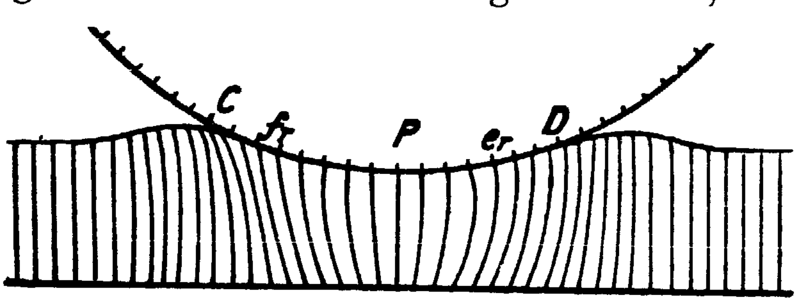


FIG. 67.

a flat thick sheet of india-rubber. The roller sinks into the rubber and has contact with it from *C* to *D*. Lines drawn on the india-rubber originally parallel and equidistant are distorted as shown. The motion of the roller being from the left to the right, contact begins at *D* and ceases at *C*. The surface of the rubber is depressed at *P*, the lowest point of the wheel, and is bulged upwards in front of, and behind, the roller. The vertical compression of the layers of the rubber below *P* causes them to bulge laterally, whilst the extension vertically of the layers in front of *D* causes them to get thinner laterally. This creates a tendency to a creeping motion of the rubber along the roller. If the resistance to sliding friction between the surfaces be great, no relative slipping may take place, but if the frictional resistance be small, slipping will take place, and energy will be expended. *e r* and *f r* limit the surfaces over which there is no slipping ; between *e r* and *D*, and again between *f r* and *C*, there is no relative slipping.

This action is such as to cause the distance actually travelled by a roller in one revolution to be different from the geometric distance. Thus, an iron roller rolled about two per cent. less per revolution when rolling on rubber than when rolling on wood or iron. The following table shows the actual slipping of a rubber tyre three-quarters of an inch thick, glued to a roller.

Nature of surface	Distance travelled in one revolution	Circumference of the ring	Amount of slipping
Steel bar	22·55 in.	22·5 in.	−0·05 in.
India-rubber 0·156 in. thick (clean)	22·55 ,,	22·5 ,,	−0·05 ,,
Ditto (black-leaded) . .	22·55 ,,	22·5 ,,	−0·05 ,,
Ditto 0·08 in. thick (clean) .	22·5 ,,	22·5 ,,	0·0 ,,
Ditto (black-leaded) . .	22·52 ,,	22·5 ,,	−0·02 ,,
Ditto 0·36 in. thick (clean) .	22·39 ,,	22·5 ,,	0·11 ,,
Ditto (black-leaded) . .	22·42 ,,	22·5 ,,	0·08 ,,
Ditto 0·75 in. thick (clean) .	22·4 ,,	22·5 ,,	0·1 ,,
Ditto (black-leaded) . .	22·4 ,,	22·5 ,,	0·1 ,,

With regard to the work lost in rolling friction, a little consideration will show that a soft substance like rubber will waste more work, and therefore have a greater rolling resistance than a harder substance such as iron or steel. Professor Osborne

Reynolds has shown that the rolling resistance of rubber is about ten times that of iron. Experiments were made on a cast-iron roller and plane surfaces of different materials, the plane being inclined sufficiently to cause the roller to start from rest. The following table shows the mean of results for various conditions of surface and manner of starting, the figures tabulated giving the vertical rise in five thousand parts horizontal.

Nature of surface	Starts from rest		Starts from rest in the opposite direction		Mean
	Clean	Oiled or black-leaded	Clean	Oiled or black-leaded	
Cast-iron . .	5·66	5·61	2·57	2·36	4·05
Glass . . .	6·32	5·96	1·93	2·56	4·19
Brass . . .	7·75	6·53	2·07	2·587	4·73
Boxwood . .	10·05	9·25	5·71	2·34	7·09
India-rubber .	35·37	38·75	31·87	28·00	33·24

CHAPTER X

STRAINING ACTION: TENSION AND COMPRESSION

79. **Action and Reaction.**—Newton's third law of motion is thus enunciated:

"To every action there is always an equal and contrary reaction; or, the mutual action of any two bodies are always equal, and oppositely directed in the same straight line; or, action and reaction are equal and opposite."

We have in the preceding chapters spoken of single forces, but remembering that force can only be exerted by the mutual action of two bodies, the truth of Newton's third law is apparent. If a rider press his saddle downwards with a force of 150 lbs., the saddle presses him upwards with an equal force; if he pull at his handles, the handles exert an equal force on his hands in the opposite direction. The passive forces thus called into existence are quite as real as what are apparently more active forces. For example, suppose a man to pull at the end of a rope with a force of 100 lbs., the other end of which is fastened to a hook in a wall, the hook exerts on the rope a contrary pull of 100 lbs. Suppose now that two men at opposite ends of the rope each exert a pull of 100 lbs., the 'active' pull of the second man in the second case is exactly equivalent to the 'passive' pull of the hook in the first case.

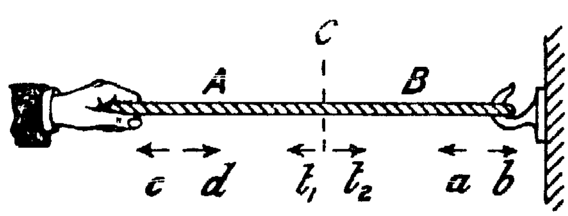


FIG. 68.

The different forces must be carefully distinguished in such cases. Thus, in figure 68 the force exerted by the rope on the hook in the wall is in the direction a, the force exerted by the hook on the rope is in the direction b, the pull exerted by the man on the end of the rope

is in the direction c, and the pull of the rope on the man is in the direction d.

80. **Stress and Strain.**—Consider the rope divided at C into two parts, A and B. The part A will exert a pull in the direction t_1 on B, and similarly the part B will exert a pull in the direction t_2 on A. The two forces t_1 and t_2 constitute a straining action at C.

In the case of a rope the forces b and c acting on its ends are directed outwards, and the straining action is called a *tension*.

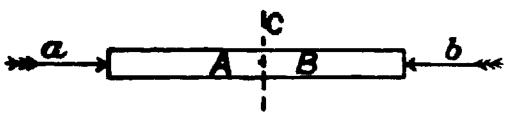


FIG. 69.

If a bar (fig. 69) be subjected to equal forces, a and b, at its ends acting inwards, the straining action is called a *compression*.

In figures 68 and 69 the parts A and B tend to separate from or approach each other in a direction at right angles to the plane C. If the parts A and B tend to slide relative to each

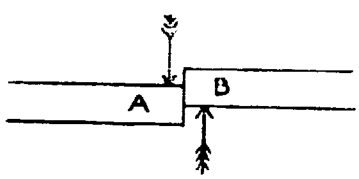


FIG. 70.

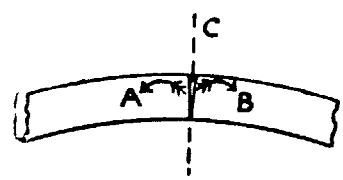


FIG. 71.

other in the direction of the plane (fig. 70), the straining action is called *shearing*.

If the parts A and B tend to rotate about an axis perpendicular to the axis of the bar (fig. 71), the straining action is called *bending*.

If the parts A and B tend to rotate in opposite directions about the axis of the bar (fig. 72), the straining action is called *torsion*.

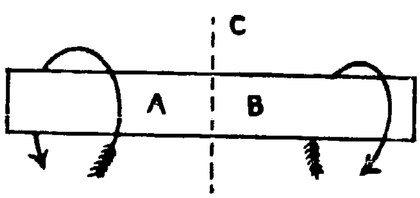


FIG. 72.

Compound straining actions consisting of all or any of the simple straining actions may take place.

These straining actions are resisted by the mutual action between the particles of the material, this mutual action constituting the *stress* at the point.

Tensile Stress.—If a bar be subjected to forces as in figure 68, every transverse section throughout its length is subject to a *tensile*

stress. If P be the magnitude of the forces b and c (fig. 68), and A the area of the transverse section at C, the force acting on each unit of transverse section—that is, the tensile stress per unit of area—is

$$p = \frac{P}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Compressive Stress.—In the same way, if the bar be subjected to forces directed inwards (fig. 69), every transverse section of it is subjected to a *compressive stress.* The compressive stress per unit of area will also be in this case

$$p = \frac{P}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

81. **Elasticity.**—If a bar of unit area (fig. 73) be fixed at one end, and subjected at the other end to a load, p, it is found that its length is increased by a small quantity. If the load does not exceed a certain limit, when it is removed the bar recovers its original length. It is found experimentally that with nearly all bodies, metals especially, this increase in length, x, is proportional to the load, and to the original length of the bar, so that we may write

$$x = \frac{p\,l}{E}$$

or,

$$\frac{p}{E} = \frac{x}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

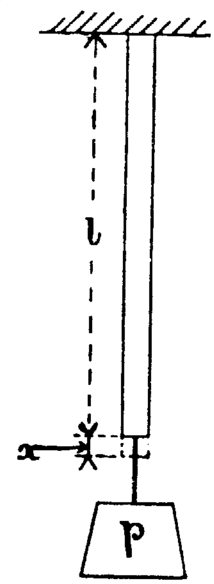


FIG. 73.

where E is a constant, the value of which depends on the nature of the material. The ratio of this elongation to the original length—that is, the extension per unit of length—is called the *extension*, and denoting it by e we have

$$e = \frac{x}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

substituting in (2) we have

$$\frac{p}{e} = \mathrm{E} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

E is called the *modulus of elasticity* of the material. A general idea of its nature may be had as follows: Conceive the material to be infinitely strong, and to stretch under heavy loads at the same rate as under small loads. Let the load be increased until the change of length, x, is equal to l, the original length of the bar. Substituting $x = l$ in (2) we have $p = E$. That is, the modulus of elasticity is the stress which would be required to extend the bar to twice its original length, provided it remained perfectly elastic up to this limit.

The value of E for cast iron varies from 14,000,000 to 23,000,000 lbs. per sq. in.; for wrought-iron bars, from 27,000,000 to 31,000,000 lbs. per sq. in.; for steel plate 31,000,000 lbs. per sq. in.; for cast steel, tempered, 36,000,000 lbs. per sq. in.

Example.—The spokes of a wheel are No. 16 W.G., 12 inches long; the nipples are screwed up till the spokes are stretched $\frac{1}{100}$ in. What is the pull on each spoke?

Taking $E = 36{,}000{,}000$ lbs. per sq. in., and substituting in (2), we get

$$\frac{p}{36{,}000{,}000} = \frac{\frac{1}{100}}{12},$$

from which,

$$p = 30{,}000 \text{ lbs. per sq. in.}$$

A, the sectional area of each spoke (Table XII., p. 346), is ·00322 sq. in.; P, the total pull on the spoke, is $p\,A$. Therefore,

$$P = 30{,}000 \times {\cdot}00322 = 96{\cdot}6 \text{ lbs.}$$

82. **Work done in Stretching a Bar.**—In section 81 we have found the stress, p, corresponding to an extension, x, of a bar; we can now find the work done in stretching the bar. It will be convenient to draw a diagram to represent graphically the relation between p and x. Let $A\,B_0$ (fig. 74) be the bar, fixed at A, and let B_0 be the position of the lower end when subjected to no load. Under the action of the load P let the lower end be stretched into position B, then $B_0\,B = x$. Let $B\,N$ be drawn at right angles to the axis of the bar, representing to any convenient scale the load P. If these processes be repeated for a

number of different values of P, the locus of the point N will be a straight line passing through B_0, and the area of the triangle $B_0 B N$ will represent the work done in stretching the bar the distance $B_0 B$. Therefore,

$$\text{Work done} = \tfrac{1}{2} P x \quad . \quad . \quad . \quad (5)$$

Substitute the value of x from (2) in (5), and remembering that $P = Ap$, we get

$$\text{Work done} = \frac{p^2}{E} \cdot \frac{Al}{2} = \frac{p^2}{2E} \times \text{volume of the bar} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

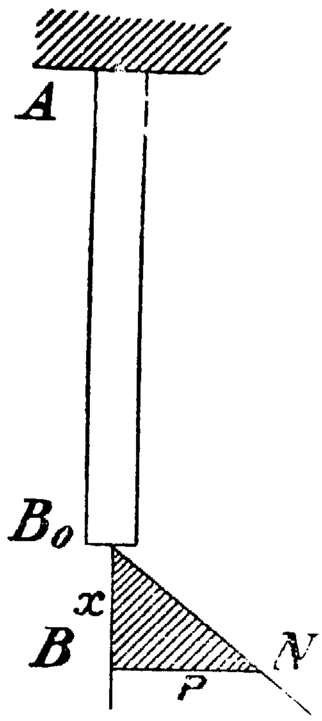


FIG. 74.

Therefore the quantities of work done in producing a given stress, p, on different bars of the same material are proportional to the volumes of the bars. On bars of equal volume but of different materials the quantities of work done in producing a given stress, p, are inversely proportional to the moduli of elasticity. The work done in stretching a given bar is proportional to the square of the stress produced.

If the bar be tested up to its elastic limit, f, the work done is

$$\frac{f^2}{2E} \times \text{volume of bar.}$$

This gives a measure of the work that can be done on the bar without permanently stretching it. The quantity $\frac{f^2}{2E}$ depends only on the material, is called its *modulus of resilience*, and gives a convenient measure of the value of the material for resisting impact or shock.

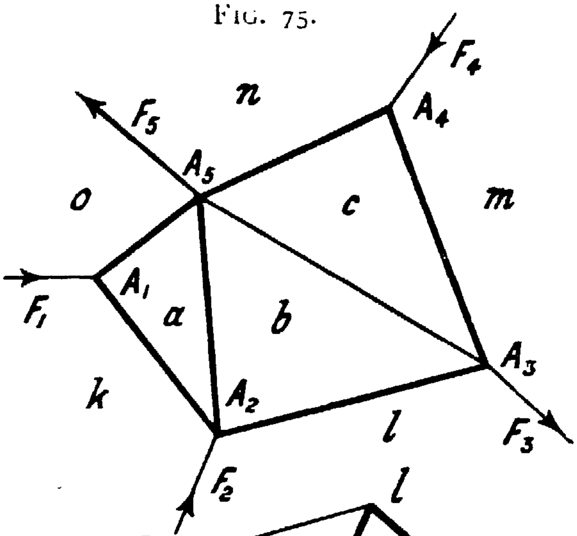
Example.—The work done in stretching the spoke in the example, section 81, is

$$\tfrac{1}{2} \cdot \times 96{\cdot}6 \times \tfrac{1}{100} = {\cdot}483 \text{ inch-lb. or } {\cdot}04 \text{ foot-lb.}$$

83. **Framed Structures.**—A framed structure is formed by jointing together the ends of a number of bars by pins in such a manner that there can be no relative motion of the bars without

distorting one or more. If each bar be held at only two points, and the external forces be applied at the pins, the stress on any bar must be parallel to its axis, and there will be no bending. In figure 75 let the external forces F_1, F_2 . . . be applied at the pins A_1, A_2 . . . Let the frame be in equilibrium under the forces, and let F_1, F_2 . . . (fig. 76) be the sides of the force-polygon. If all the forces F_1, F_2 . . . be known, it will be possible, in general, to find the stress on each bar of the frame by a few applications of the principle of the force-triangle. In a trussed beam (*e.g.* a bridge, roof, or bicycle frame) the external forces are the loads carried by the structure, whose magnitude and lines of action are generally known, and the reactions at the supports. If there are two supports the reactions can be determined by the methods of section 17, so that they shall be in equilibrium with the loads.

FIG. 75.



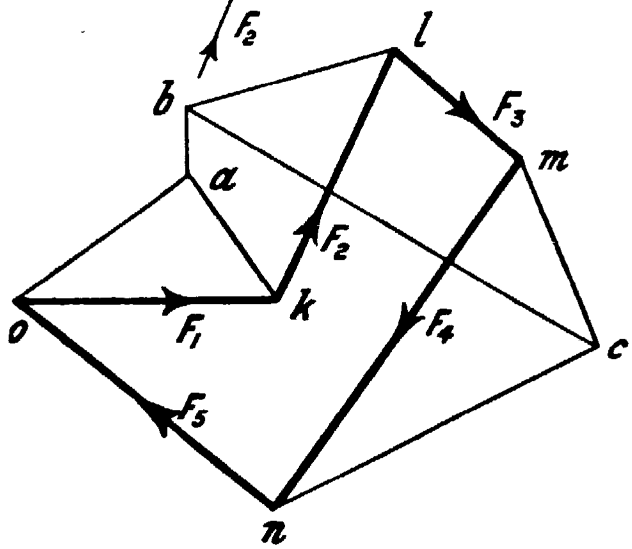


FIG. 76.

To find the stresses on the individual members of the frame we begin by choosing a pin at which two bars meet and one external load acts; the magnitude and direction of the latter, and the direction of the forces exerted by the bars on the pin, being known, the force-triangle for the pin can be drawn. Thus, beginning at the pin A_1, on which three forces (the external force F_1, and the thrusts of the bars A_1 A_2 and A_1 A_5) act, the force-triangle can be at once drawn. Before proceeding with this drawing it will be convenient to use the following notation: Let the spaces into which the bars divide the frame be denoted by a, b, . . . , and the spaces between the external forces F_1, F_2, . . .

by k, l, . . . , then the bar $A_1 A_2$ which divides the spaces a and k will be denoted by $a\ k$, the stress on this bar will also be denoted by $a\ k$. The force-triangle for the pin A_1, at which point the spaces a, o and k meet, is $a\ o\ k$ (fig. 76). Proceeding now to the pin A_2, at which four forces act, the external force F_2 and that exerted by the bar $a\ k$ are known, and the direction of the forces exerted by the bars $a\ b$ and $b\ l$ are known. Two sides, $a\ k$ and $k\ l$, of the force-polygon for the pin A_2 are already drawn, the polygon is completed by drawing $a\ b$ and $l\ b$ (fig. 76) respectively, parallel to the bars $a\ b$ and $l\ b$ (fig. 75). Proceeding now to the pin A_3, only two forces are as yet unknown, and of the force-polygon two sides, $b\ l$ and $l\ m$, are already drawn. The remaining sides, $b\ c$ and $m\ c$, are drawn parallel to the corresponding bars (fig. 75). At the pin A_5, four of the forces acting are already known, and the corresponding sides, $n\ o$, $o\ a$, $a\ b$, and $b\ c$, of the force-polygon are already drawn. The side $n\ c$ of the force-diagram must therefore be parallel to the corresponding bar of the frame-diagram, and a check on the accuracy of the drawing is obtained.

With the above notation, the letters $A_1\ A_2$ . . . and $F_1\ F_2$ . . . may be suppressed.

Figure 75 is called the *frame-diagram* and figure 76 the *stress-diagram*, or *force-diagram*. In the force-diagram, the polygon of external forces is drawn in thick lines, and the direction of each force is indicated by an arrow. From these arrows it will be easy to determine whether the stress on any member of the frame is tensile or compressive.

The total force on any member of a framed structure being obtained, its sectional area can be obtained at once by formula (1).

84. **Thin Tubes subjected to Internal Pressure.** An important case of simple tension is that of a hollow cylinder subjected to fluid pressure; *e.g.* the internal shell of a steam boiler, or the pneumatic tyre of a cycle wheel. In long cylindrical boilers the flat ends have to be made rigid in order to preserve their form under internal pressure, while the cylindrical shell is in stable equilibrium under the action of the internal pressure. A pneumatic tyre of circular section is also of stable form under internal pressure; a deformation by external pressure at any point will

be removed as soon as the external pressure at the point be removed.

Let p be the internal pressure in lbs. per sq. in., d the diameter and t the thickness of the tube (fig. 77). Consider a section by a plane, $A\ A$, passing through the axis of the tube. The upper half, $A\ B\ A$, is under the action of the internal pressure p, distributed over its inner surface, and the forces T due to the pull of the lower part of the tube; therefore $2\ T =$ the resultant of pressure p on the half tube. This resultant can be easily found by the following artifice: Consider a stiff flat plate joined at $A\ A$ to the half tube, so as to form a **D** tube. If this tube be subjected to internal pressure, p, and to no external forces, it must remain at rest; if otherwise, we would obtain perpetual motion. Therefore, the resultant pressure R_1 on the curved part must be equal and opposite to the resultant pressure R_2 on the flat portion of the tube. If we consider a portion of the tube 1 in. long in the direction of the axis,

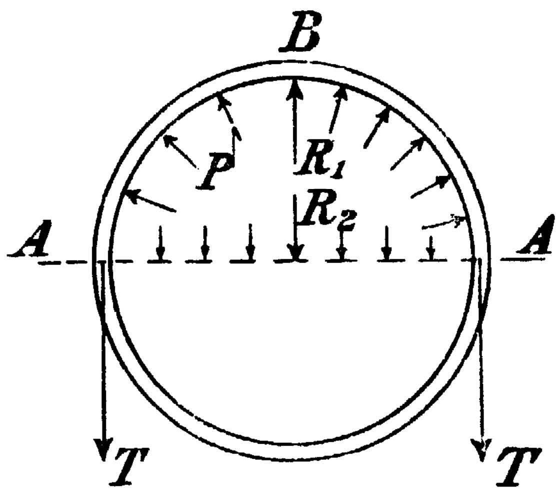


FIG. 77.

$$R_2 = p\ d,$$

and therefore

$$2\ T = p\ d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

But if f be the intensity of the tension on the sides of the tube, $T = f\ t$

$$\therefore\ 2\ f\ t = p\ d, \text{ or } f = \frac{p\ d}{2\ t} \quad . \quad . \quad . \quad . \quad (8)$$

Example.—A pneumatic tyre $1\frac{3}{4}$ in. inside diameter, outer cover $\frac{1}{16}$ in. thick, is subjected to an air pressure of 30 lbs. per square inch. The average tensile stress on the outer cover is

$$f = \frac{30 \times 1{\cdot}75}{2 \times \frac{1}{16}} = 420 \text{ lbs. per sq. in.}$$

CHAPTER XI

STRAINING ACTIONS: BENDING

85. **Introductory.**—We have in chapter x. considered the stresses on a bar acted on by forces parallel to its axis. We now proceed to consider the stresses on a bar due to forces the lines of action of which pass through the axis, but do not coincide with it. Each force may be resolved into two components, respectively parallel to, and at right angles to, the axis. The components parallel to the axis may be treated as in the previous chapter. Of the transverse forces, the simplest case is that in which they all lie in the same plane, a beam supporting vertical loads being a typical example. Such a beam must be acted on by at least three forces, the *load* and the two *reactions* at the supports.

86. **Shearing-force on a Beam.**—If a bar in equilibrium be acted on by three parallel forces at right angles to its axis (fig. 78), every section by a plane parallel to the direction of the forces will be subjected to a bending stress.

Consider the body divided into two portions by a plane at X. Under the action of the force R_1 the part A will tend to move upwards relative to the part B. The part A therefore acts on the part B with a force R_1' equal and parallel to R_1, and the part B reacts on the part A with an equal opposite force R_1''. The two forces R_1' and R_1'' at X constitute a shearing at the section. It will easily be seen that the shearing-force will be the same for all sections of the beam between the points of application of the forces R_1 and W, and that the shearing-force on the section X_1 will be the algebraic sum of the forces to the left-hand side, or to the right-hand side, of the section. This is true for a beam acted on by any number of parallel forces.

In particular, if a beam be supported at its ends (fig. 78) and

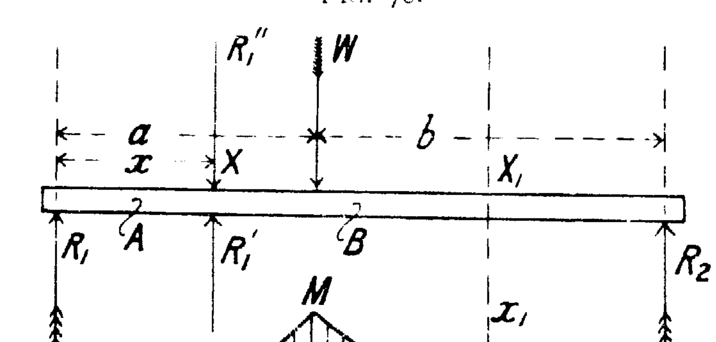
loaded with a weight, W, the reactions R_1 and R_2 at the supports will, by section 49, be equal to

$$\frac{bW}{a+b}, \quad \frac{aW}{a+b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where a and b are the segments in which the length of the beam is divided at the point of application of the load. The shearing-force on the part A will be equal to R_1, and the shearing-force on the part B will be equal to $R_1 - W = - R_2$.

Shearing-force Diagram.—The value of the shearing-force at any section of a beam is very conveniently represented by drawing an ordinate of length equal to the shearing-force at the corresponding section, any convenient scale being chosen. The shaded figure (fig. 79) is the shearing-force diagram for a beam supported at the ends and loaded with a single weight.

FIG. 78.



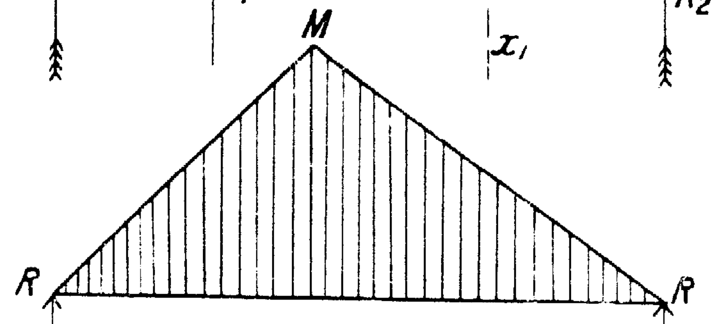


FIG. 80.

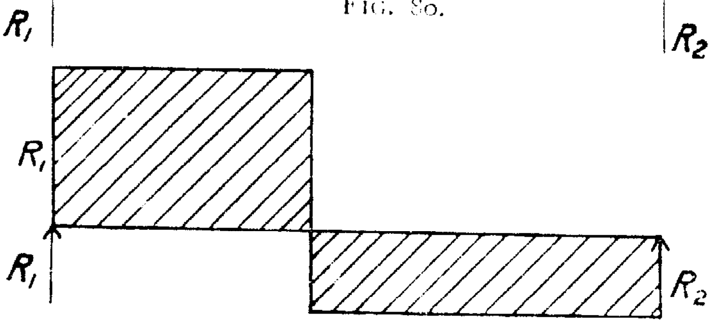


FIG. 79.

The shearing-force at the section X (fig. 78) is of such a nature that the part on the left-hand side tends to slide *upwards* relative to the part on the right-hand side of the section. The shearing-force at X_1 is of such a nature that the part on the left tends to slide *downwards* relative to the part at the right of the section. Thus shearing-forces may be opposite in sign; if that at X be called positive, that at X_1 will be negative. The diagram (fig. 79) is drawn in accordance with this convention.

87. **Bending-moment.**—If a bar of length, l, be fixed horizontally into a wall (fig. 83), and be loaded at the other end with a

weight W, the said weight will tend to make bar turn at its support, the tendency being measured by the moment Wl of the force. This tendency is resisted by the reaction of the wall on the beam. The section of the beam at the support is said to be subjected to a *bending-moment* of magnitude Wl.

From this definition a weight of 50 lbs. at a distance of 20 inches will produce the same bending-moment as a weight of 100 lbs. at a distance of 10 inches; the bending-moment being 50×20, or $100 \times 10 = 1000$ inch-lbs.

Returning to the discussion of figure 78, it will be seen that the part A is acted on by two equal, parallel, but opposite forces, R_1 and R_1'', constituting a couple of moment $R_1 x$ tending to turn the part A. But the part A is actually at rest; it must, therefore be acted on by an equal and opposite couple. The only other forces acting on A are those exerted by the part B at the section X. The upper part of portion B (fig. 81, which is part of figure 78 enlarged) acts on the portion A with a number of forces, c_1 c_1, diminishing in intensity from the top towards the middle of the beam; the resultant of these may be represented by C_1. The lower part of B acts on A with the forces t_1 t_1, whose resultant may be represented by T_1. Since the part A is in equilibrium, the resultant of all the horizontal forces acting on it must be zero; therefore T_1 and C_1 are equal in magnitude, and constitute a couple which must be equal to $R_1 x$. If d be the distance between T_1 and C_1, we must therefore have

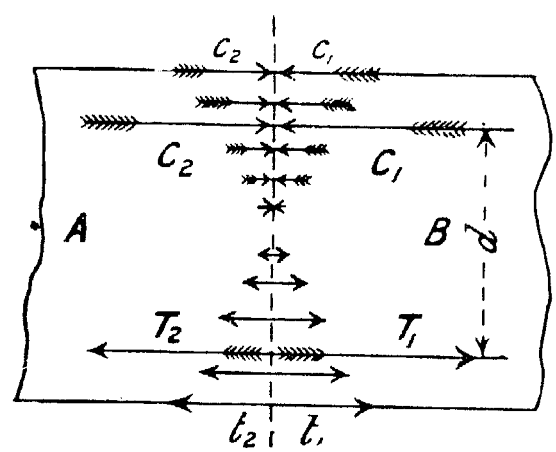


FIG. 81.

$$T_1 d = R_1 x.$$

The part A acts on the part B with forces c_2 at the top, and forces t_2 at the bottom of the beam; the resultants being indicated by C_2 and T_2 respectively. The two sets of forces c_1 and c_2 constitute a set of compressive stresses on the upper portion of the beam at X, and the two sets of forces t_1 and t_2 constitute a set of tensile

stresses on the lower portion of the beam. The moment of the couple $R_1 . x$ is called the bending-moment at the section X; while the moment of the couple $T_1 d$ is called the moment of resistance of the section.

The existence of the shearing-force and bending-moment at any section of a beam can be experimentally demonstrated by actually cutting the beam, and replacing by suitably disposed fastenings the molecular forces removed by the cutting. Figure 82 shows diagrammatically a cantilever treated in this manner. The shearing-force at the section is replaced by the upward pull W of a spiral spring, and the couple acting on the part B formed by the load W, and the pull of the spring is balanced by the equal and opposite couple formed by the pull T_2 of the fastening bands at the top and the thrust C_2 of the short strut at the bottom of the section.

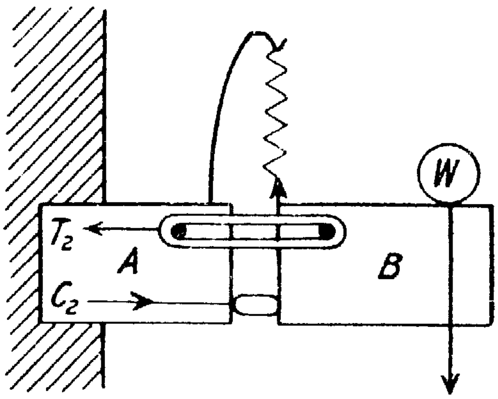


FIG. 82.

Bending-moment Diagram.—The bending-moment at any section of a beam can be conveniently represented by a diagram, the ordinate being set up equal in length to the bending-moment at the corresponding section.

Since the bending-moment at the section X (fig. 78) is the product of the force R_1 into the distance x of the section from its point of application, the further the section X be taken from the end of the beam the greater will be the bending-moment. In the case of a beam supported at the ends and loaded at an intermediate point with a weight W, the bending-moment M on the section over which W acts will be given by—

$$M = R_1 a = \frac{a\,b}{a+b} W \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the bending-moment on any section between R_1 and W will be represented by the ordinate of the shaded area in figure 80.

The bending-moment at the section X_1 (fig. 78) is the sum of the moments of the forces R_1 and W about X_1; or is equal to the moment of the force R_2 about X_1.

88. **Simple Examples of Beams.**—A few of the most commonly occurring examples of beams may be discussed here. Figure 83 shows a cantilever of length, l, supporting a weight, W, at its end. The bending-moment at a section very close to the support is Wl, that at a section distant x from the outer end of the cantilever is Wx. The bending-moment diagram is, therefore, a straight line, the maximum ordinate, Wl, being at the support, that at the end zero. The shearing-force is equal to W for all sections; the shearing-force diagram is, therefore, a straight line parallel to the axis.

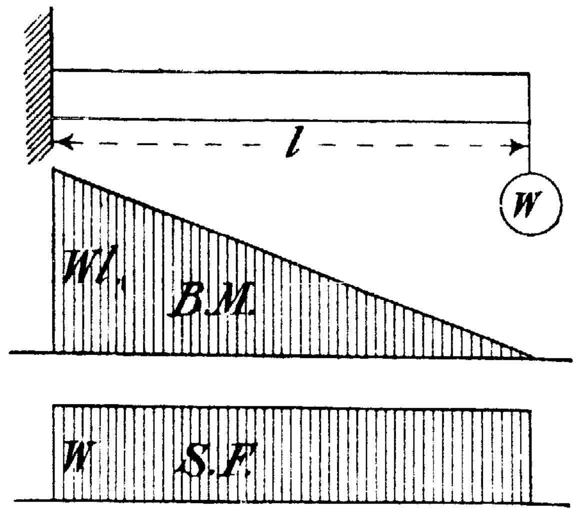


FIG. 83.

Figure 84 shows a cantilever loaded uniformly, the total weight being W. The resultant weight acts at the middle of the cantilever distant $\frac{l}{2}$ from the support, the bending-moment at the support is, therefore, $\frac{Wl}{2}$. At any section distant x from the end of the cantilever, we find the bending-moment as follows: Consider the portion of the cantilever lying to the right of the section, the resultant of the load resting on it is wx, w being the weight per unit of length, and acts at a distance $\frac{x}{2}$ from the section. The bending-moment on the section is therefore

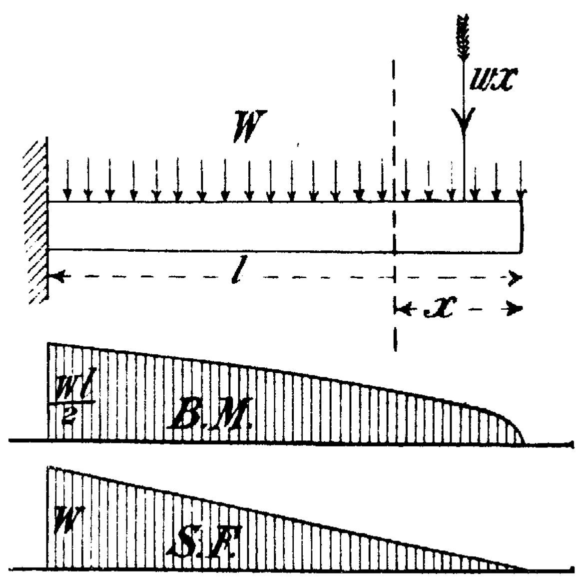


FIG. 84

$$M = \frac{w x^2}{2} = \frac{W x^2}{2 l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Plotting these values for different values of x, the bending-moment curve is a parabola.

The shearing-force on the section distant x from the end is $w\,x = \frac{Wx}{l}$. Plotting these values for different values of x, we get the shearing-force curve a straight line, having the ordinate W at the support and zero ordinate at the end.

Figure 85 shows a beam of span, l, supporting a load, W, at the middle. The reactions at the support are evidently each equal to $\frac{W}{2}$, the bending-moment at any section distant x from the end is therefore $\frac{Wx}{2}$, x being less than $\frac{l}{2}$. At the middle of the beam the bending-moment is a maximum, and equal to

$$\frac{W}{2} \cdot \frac{l}{2} = \frac{Wl}{4} \quad . \quad . \quad (4)$$

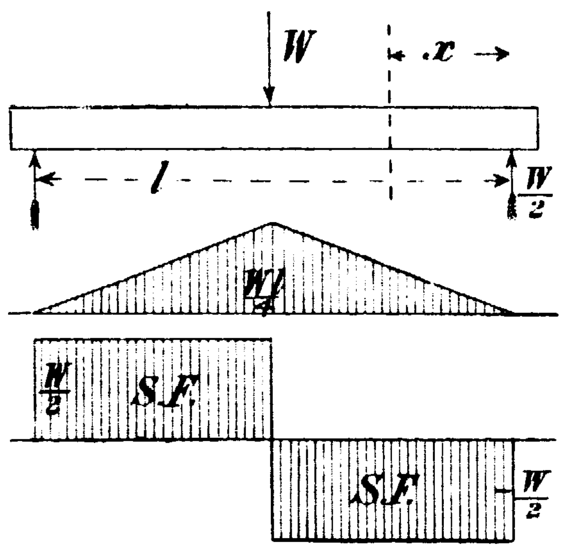


FIG. 85.

The bending-moment curve is a triangle, the maximum ordinate being in the middle. The shearing-force is constant and equal to $\frac{W}{2}$ from one end up to the middle of the beam, then changes sign and becomes $-\frac{W}{2}$ over the other half.

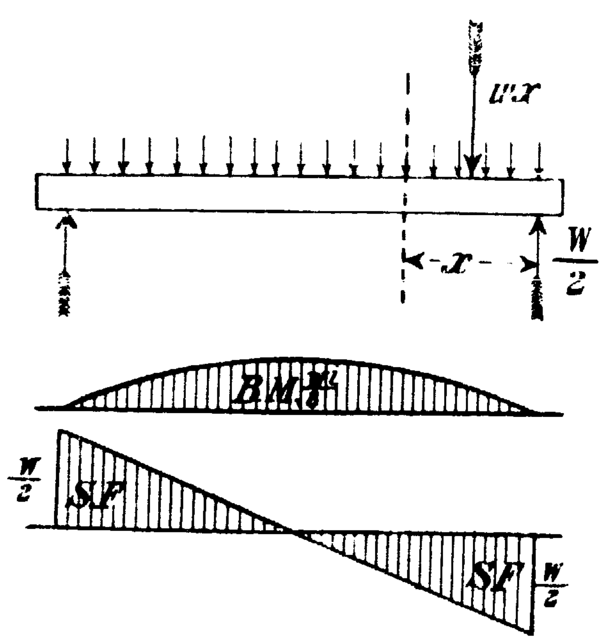


FIG. 86.

Figure 86 shows a beam supporting a load, W, uniformly distributed. The reaction at each support is evidently $\frac{W}{2}$; the bending-moment at a section distant x from the end is the sum of the moments due to the reaction $\frac{W}{2}$, and of the resultant load $w\,x$ acting on the

right-hand side of the action at a distance $\frac{x}{2}$ from the section,

$$\therefore\ M = \frac{W}{2}x - w\,x\,.\,\frac{x}{2} = \frac{W}{2}\left(x - \frac{x^2}{l}\right) \quad . \quad . \quad (5)$$

If x be made equal to $\frac{l}{2}$, the above formula gives the bending-moment at the middle of the beam, $M_0 = \frac{W}{2}\left(\frac{l}{2} - \frac{l}{4}\right) = \frac{Wl}{8}$. The bending-moment curve is a parabola with its maximum ordinate $\frac{Wl}{8}$ at the middle of the beam.

89. **Beam supporting a Number of Loads at Different Points.**—The loads and their positions along the beam being given, the reaction R_1 at one support can be found by taking moments about the other support; the bending-moment at any section can then be calculated by adding algebraically the moments of all the forces on either one side or other of that section. The reactions R_1 and R_2 at the supports can also be found by the method of sections 47 and 48. Since in this case the forces are all parallel, the construction is simplified; the force-polygon becomes a straight line, and the corners of the link-polygon lie on the vertical lines of action of the loads and reactions.

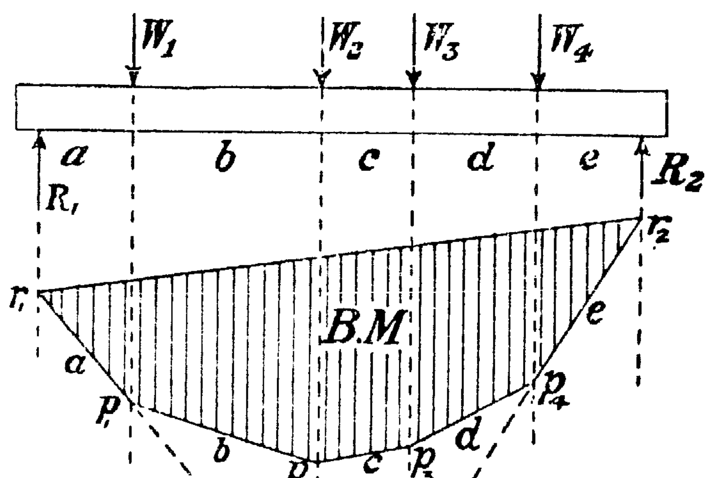
Figure 87 shows a beam supporting a number of weights, W_1, W_2, W_3, W_4, and figure 88 the force-polygon a, b, c, d, e. The construction of figure 41 becomes as follows: From any point p_1 on the line of action of W_1 draw a straight line b parallel to the line Ob (fig. 88). From p_2, where this line cuts the line of action of W_2, draw a straight line, c, parallel to the line Oc; continuing this process until the point p_4 on the line of action W_4 is reached.

Through p_1 and p_4 draw $p_1\,p_r$ and $p_4\,p_r$ parallel to Oa and Oe respectively, intersecting each other at p_r and the lines of action of R_1 and R_2 at r_1 and r_2 respectively. The resultant of the loads W_1, W_2, W_3 and W_4 passes through p_r. Through O draw Or parallel to $r_1\,r_2$; then the reactions R_1 and R_2 are equal to ra and er respectively.

Link-polygon as Bending-moment Diagram.—If the pole O be chosen at random, the closing line $r_1\,r_2$ of the link-polygon will

not, in general, be parallel to the axis of the beam. Let a new pole, O^1, be taken by drawing $O\,O^1$ parallel to, and $r\,O^1$ at right angles to, the lines of action of the loads W_1, W_2 . . ., and let a

FIG. 87.



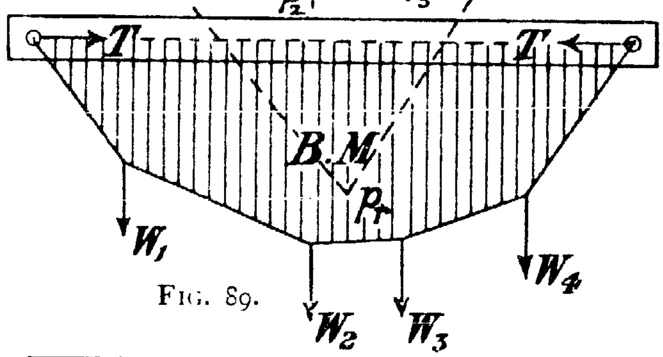


FIG. 89.

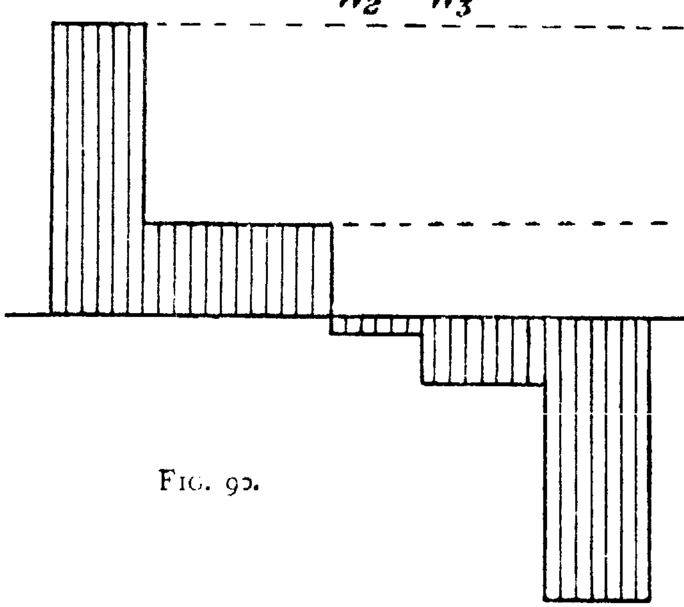


FIG. 90.

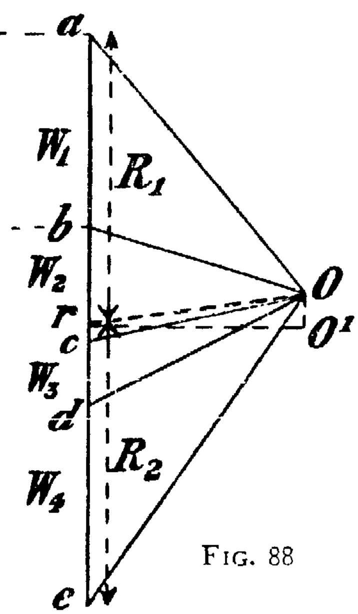


FIG. 88

new link-polygon (fig. 89) be drawn. If a thin wire be made to the same outline as this same polygon and be attached to the beam, and the loads W_1, W_2 . . . attached at the angles, it is evident that the compound structure formed by the bar and

wire is subjected to the same bending stresses as the beam (fig. 87). In both cases the dispositions of the loads and reactions are identical; but in the compound structure the bar is subjected to a thrust, T, represented in the force-diagram (fig. 88) by $O^1 r$. Considering the corner of the wire at which W_1 acts, the tensions on the two portions of the wire, and the force W_1 are in equilibrium, and are represented by the force-triangle $O^1 a b$ (fig. 88); similarly for the other portions of the wire. It will be noticed that at each part of the wire the horizontal component of the pull is equal to $O^1 r$; that is, equal to T. Taking any vertical section of the compound structure (fig. 89) the mutual actions consist of a thrust, T, on the bar, an equal horizontal pull, T, on the wire, and the vertical component of the pull on the wire. The two former constitute the bending-couple at the section, the latter the shearing-force. The bending-moment on any section of the beam is therefore equal to $T h$, h being the ordinate of the link-polygon; the link-polygon can therefore be used as a bending-moment diagram.

The shearing-force on any section of the beam (fig. 87) is equal to the vertical component of the pull on the wire (fig. 89), which is equal to the vertical component of the corresponding line from the pole O^1 (fig. 88). A shearing-force diagram (fig. 90) can therefore be constructed by projecting over a base line from r, and straight lines from a, b . . . (fig. 88) to the corresponding divisions of the beam.

Example.—Calculate, and draw, a bending-moment diagram for the frame of a tandem bicycle carrying two riders, each 150 lbs. weight (30 lbs. of which is assumed to be applied at the crank-axle); the wheel-base being 64 inches long, the rear crank-axle being 19 inches in front of the rear wheel centre, the crank-axles 22 inches apart, and the saddles 10 inches behind their respective crank-axles.

The figures of illustrations are given in chapter xxiii., page 327.

To calculate the reactions on the wheel spindles, take moments about the centre of the rear wheel—

$$(120 \times 9) + (30 \times 19) + (120 \times 31) + (30 \times 41) - (R \times 64) = 0$$

from which, $R_1 = 103{\cdot}1$ lbs.,

and $R_2 = 196{\cdot}9$ lbs.

The greatest bending-moment, which occurs on the vertical section passing through the front seat, is

$$M = (103\cdot1 \times 33) - (30 \times 10) = 3{,}102 \text{ inch-lbs.}$$

The frame, or beam (fig. 321) is drawn $\frac{1}{32}$nd full size; the scale of the force-diagram (fig. 323) is 1 inch to 400 lbs., and the pole distance O^1 corresponds to 125 lbs.; 1 inch ordinate of the bending-moment diagram (fig. 324) therefore represents 32 in. × 125 lbs., *i.e.* 4,000 inch-lbs.

The results got by the graphical and arithmetical methods must agree; thus a check on the accuracy of the work is obtained.

90. **Nature of Bending Stresses.**—We must now consider more minutely the nature of the stresses t and c (fig. 81) on any section subject to bending.

Let a beam be acted on by two equal and opposite couples at its ends; it will be bent into a form, shown greatly exaggerated in figure 91. It can be easily seen that the bending-moment on the middle portion of the beam will be of the same value throughout, and if the section is uniform, the amount of bending will be the same at all sections; that is, the beam, originally straight, will be bent into a circular arc.

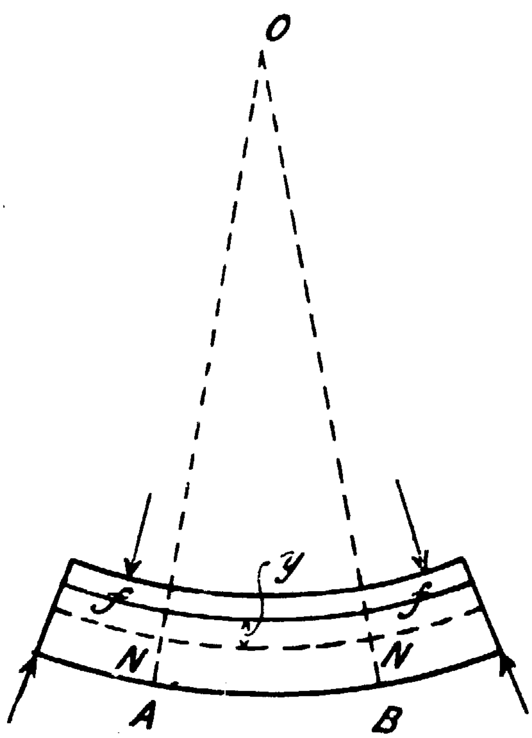


FIG. 91.

Consider the portion of the beam included between two parallel sections A and B. After bending, these sections are inclined, and if produced, will meet at the centre of curvature of the beam. The top fibres of the beam will be shortened and the lower fibres lengthened, while those at some intermediate layer, NN, will be unaltered in length. The surface in which the centres of the fibres NN lie is called the *neutral surface* of the beam, while its line of intersection with a transverse plane is called the *neutral axis* of the section. Now, suppose that the fibres could be laid out flat and

of exactly the same length as they are after bending. If the left-hand ends all lay in the plane $A\ A$ (fig. 92) at right angles to $N N$, the other ends must evidently lie in a plane $B^1\ B^1$; $B\ B$ representing the plane in which the ends of the unstretched fibres would lie. The distance, parallel to $N N$, included between the

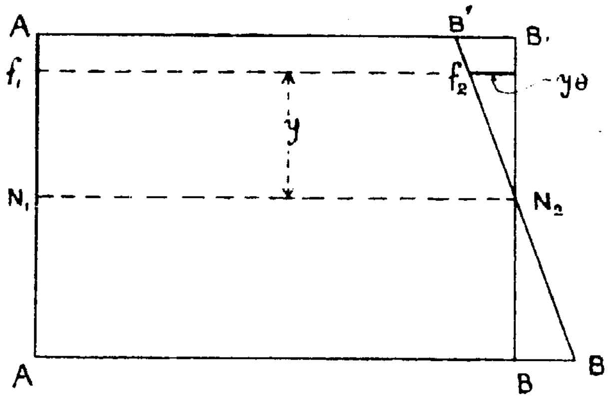


FIG. 92.

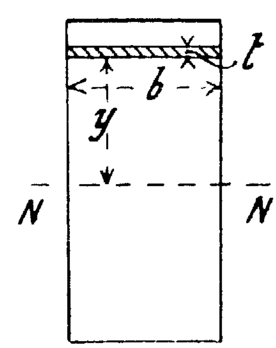


FIG. 93.

lines $B\ B$ and $B^1\ B^1$ gives the amount of the contraction or elongation of the corresponding fibres. The elongation or contraction of any fibre is thus seen to be proportional to its distance from $N N$. Now the stress on a bar or fibre is proportional to the extension produced; therefore the stress on the fibres of a beam varies as the distance from the neutral axis.

Let O be the centre, and R the radius of curvature of $N N$ (fig. 91), y the distance of any fibre f above the neutral axis, θ the angle $N O N$ subtended at the centre O by the portion of the fibre considered. The radius of curvature of the fibre f is $(R-y)$, the length of the arc $f_1\ f_2$ (fig. 92) is therefore $(R-y)\ \theta$; and the length of the arc $N_1\ N_2$ is $R\ \theta$. A fibre at the neutral axis is unaltered in length by bending, so the length $N_1\ N_2$ is the same as in the straight position. The length of the fibre $f_1\ f_2$ was originally equal to that of $N_1\ N_2$; the decrease in its length is therefore

$$R\ \theta - (R-y)\ \theta = y\ \theta\ ;$$

its compression per unit of length is therefore

$$\frac{y\ \theta}{R\ \theta} = \frac{y}{R}.$$

By section 81, the stress producing this compression is

$$p = \frac{E\ y}{R} \quad \ldots\ldots\ldots\ldots \quad (6)$$

That is, the intensity of stress on any fibre of a beam subject to bending is proportional to its distance from the neutral axis, and inversely proportional to the radius of curvature of the neutral axis. If a fibre below the neutral axis be taken, y will be negative, the fibre will be stretched, and the stress on it will be tensile.

Since the material near the neutral axis is subjected to a low stress, it adds very little to the strength of the beam, while it adds to the weight. It is therefore economical to place the material as far as possible from the centre of the section. The framework of the earliest bicycles was made of solid bars; but a great saving of weight, without sacrificing strength, was effected by using hollow tubes. The same principle is carried out to a fuller extent in a well-designed Safety frame; the top- and bottom-tubes together forming a beam, in which practically all the material is at a great distance from the neutral axis. If the frame be badly designed, however, the top- and bottom-tubes may form merely two more or less independent beams, instead of one very deep beam.

91. **Position of Neutral Axis.**—Consider the equilibrium of the portion of the beam to the left hand of section A (fig. 91). There are no external horizontal forces acting on this portion, and therefore the resultant of the horizontal forces due to the internal reaction of the particles at the section A must be zero.

Let figure 93 be the transverse section at A (fig. 91), $N N$ being the neutral axis. The part of the section above $N N$ is subjected to compression, that below $N N$ to tension; the resultant compressive force must therefore be equal to the resultant force of tension. Consider a strip of the section of breadth b, and thickness t, at a distance y from the neutral axis; the area of this strip is $b\ t$, the stress per square inch is $\frac{E\,y}{R}$; the total force on it is therefore

$$\frac{E}{R} \cdot b\,t\,y.$$

The total force on the whole section will be the sum of the forces on all such strips; compression being considered positive and

tension negative. $\frac{E}{R}$ is the same for all the strips, therefore the resultant force on the section may be written

$$\frac{E}{R} \Sigma b t y,$$

$\Sigma b t y$ indicating the sum of all the products $b t y$. Since the resultant force on the section is zero, we must have

$$\Sigma b t y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Referring to section 50, it will be seen that this condition is equivalent to saying that the neutral axis must pass through the mass-centre of the section.

92. **Moment of Inertia of an Area.**—In figure 93, $b t$ is the area of a narrow strip parallel to, and distant y from, the axis $N N$; $b t y^2$ is therefore the product of a small element of area into the square of its distance from the axis. The sum of such products for all the elementary strips into which the given area can be divided is called the *moment of inertia of the area*, and, as shall be shown in the next section, is of fundamental importance in the theory of bending.

The calculation of moments of inertia for areas of given shape is beyond the scope of an elementary work like the present; a few of the most important results will be given for convenience of reference.

Let I denote the moment of inertia about an axis passing through the mass centre. Then, for a square of side h,

$$I = \frac{1}{12} h^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

For a circle of diameter d,

$$I = \frac{\pi}{64} d^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For a rectangular section of breadth b and depth h (perpendicular to the neutral axis),

$$I = \frac{1}{12} b h^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

For an elliptical section of breadth b and depth h,

$$I = \frac{\pi}{64} b h^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

For a hollow circular section of outside and inside diameters, d_1 and d_2 respectively,

$$I = \frac{\pi}{64} (d_1^4 - d_2^4) \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Let A be the area, the moment of inertia of which is being considered. Then for a rectangular section $A = b\,h$, and (10) may be written

$$I = \frac{1}{12} A h^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

For a circle $A = \frac{\pi}{4} d^2$, and (9) may be written

$$I = \frac{1}{16} A d^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Similarly, for an ellipse of breadth b and depth h, $A = \frac{\pi}{4} b\,d$; (11) may therefore be written

$$I = \frac{1}{16} A h^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

That is, for each of the three sections considered, the moment of inertia is equal to the product of the area, and the square of the depth at right angles to the axis of inertia, multiplied by a constant factor, which depends on the *shape* of the section. It can be shown that this is true for sections of all shapes, the value of the constant factor being different for different shapes of section, but the same for large or small sections of the same shape.

Moment of Inertia of an Area about Parallel Axes.—The moment of inertia of an area is least about an axis passing through the centre of area.

Let I_0 be the moment of inertia of an area A about any axis through the centre of area. Then it can be easily shown that the moment of inertia about a parallel axis distant y_0 from the centre of area is $I_0 + A\,y_0^{\,2}$.

Moment of Inertia of an Area about different Axes passing through the centre of figure.—The moment of inertia of an area

about different axes passing through the centre of figure are in general different, but however complex be the outline of the area, an ellipse can be drawn with its centre coinciding with the centre of the area, such that the moment of inertia relative to any axis drawn through the centre varies inversely as the square of the corresponding radius-vector of the ellipse. This ellipse is called the *ellipse of inertia*, or the *momental ellipse*, of the area. The axes corresponding to the major and minor axes of the ellipse are called the principal axes of the figure.

The momental ellipse for a rectangle, if drawn to a suitable scale, touches its sides. Similarly, for a triangle it can be shown that the ellipse touching the three sides at their middle points can be taken as the momental ellipse.

If the major and minor axes of the momental ellipse are equal, the ellipse becomes a circle, and the moments of inertia about all axes through the centre are equal. For example, since from symmetry the momental ellipse for a square is a circle, the moment of inertia of a square is the same for all axes passing through its centre.

93. **Moment of Bending Resistance.**—The moment about the neutral axis of all the forces p on the fibres of the cross section is called the *moment of resistance to bending* of the section, and is of course equal to the bending-moment on the section due to the external forces.

The moment of the force on the strip $b\,t$ (fig. 93) is

$$\frac{E}{R}\,b\,t\,y \times y$$

and the moment of all the forces on all the strips is

$$M = \frac{E}{R}\Sigma\,b\,t\,y^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

which may be written

$$M = \frac{E}{R}I \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Substituting the value of $\frac{E}{R}$ from (6) in (17) it may be written

$$\frac{M}{I} = \frac{p}{y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

(17) and (18) may be conveniently written together thus :

$$\frac{M}{I} = \frac{E}{R} = \frac{p}{y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

94. **Modulus of Bending Resistance of a Section.**—The greatest stress on a section occurs, as has already been shown, on the fibre furthest away from the neutral axis. Let f be this stress, then, denoting the corresponding of y by y_1, (18) may be written

$$M = \frac{I}{y_1} f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The quantity $\frac{I}{y_1}$, which is a geometrical quantity depending on the area and shape of the section, and not in any way on the material, is called the *modulus* of bending resistance of the section, and will be denoted by the letter Z. (20) may then be written

$$M = Zf \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

From (21) it is evident that the modulus of a section bears the same relation to the bending-moment on it, as the area of a section bears to the total direct tension or compression on it. The total pull on a bar is equal to the product of its area into the tensile strength per square inch. The bending-moment on any section of a beam is equal to the modulus of the section multiplied by the greatest stress on the section.

For a rectangular section

$$Z = \frac{b\,h^2}{6} = \tfrac{1}{6}\,A\,h \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

For a circular section,

$$Z = \frac{\pi}{32}\,d^3 = \tfrac{1}{8}\,A\,d \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

or approximately,

$$Z = \frac{d^3}{10} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

For a hollow circular section,

$$Z = \frac{\pi}{32}\,\frac{(d_1^4 - d_2^4)}{d_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Table III. gives the sectional areas and moduli for round bars.

From (20) and (23) it is evident that the bending-moment a round bar can resist, *i.e.* its transverse strength, is proportional to the cube of its diameter.

TABLE III.—SECTIONAL AREAS AND MODULI OF BENDING RESISTANCE OF ROUND BARS.

Diameter	Sectional area	Z	Diameter	Sectional area	Z
Inches	Sq. in.	In.^{3}	Inches	Sq. in.	In.^{3}
$\frac{1}{16}$	·0031	·000024	$\frac{13}{16}$	·5185	·0526
$\frac{1}{8}$	·0123	·000192	$\frac{7}{8}$	·6013	·0658
$\frac{3}{16}$	·0276	·000647	$\frac{15}{16}$	·6903	·0809
$\frac{1}{4}$	·0491	·001534	1	·7854	·0982
$\frac{5}{16}$	·0767	·00300	$1\frac{1}{8}$	·9940	·1398
$\frac{3}{8}$	·1104	·00517	$1\frac{1}{4}$	1·2272	·1917
$\frac{7}{16}$	·1503	·00822	$1\frac{3}{8}$	1·4849	·2552
$\frac{1}{2}$	·1964	·01227	$1\frac{1}{2}$	1·7671	·3313
$\frac{9}{16}$	·2485	·0175	$1\frac{5}{8}$	2·0739	·4211
$\frac{5}{8}$	·3068	·0240	$1\frac{3}{4}$	2·4053	·5261
$\frac{11}{16}$	·3712	·0319	$1\frac{7}{8}$	2·7611	·6471
$\frac{3}{4}$	·4418	·0414	2	3·1416	·7854

95. **Beams of Uniform Strength.**—The bending-moment on a beam generally varies from section to section along the axis; consequently, if of uniform section throughout it will be weakest where the bending-moment is greatest. A *beam of uniform strength* is one in which the section varies with the bending-moment in such a manner that the tendency to break is the same at all sections. This means that f, the maximum stress on the section, has the same value throughout, and therefore that M is proportional to Z.

For a thin hollow tube of constant external diameter throughout its length, Z is approximately proportional to the thickness; therefore for a tubular beam in which the bending-moment varies continuously the thickness should also vary continuously, if the beam is required to be of uniform strength. For example, the bending-moment on the handle-bar of a bicycle, due to the pull of the rider, increases from zero at the end to its maximum value at the handle-pillar. If the external diameter of the handle-bar be the same throughout, the lightest possible bar would vary in

thickness from the middle to the ends. This ideal handle-bar cannot be conveniently made, but an approximation thereto is sometimes made by inserting a liner at the middle, where the bending-moment is greatest ; there will in this case be three weak sections, the middle section and those just beyond the ends of the liners.

96. **Modulus of Circular Tubes.**—On account of the extensive use of tubes in bicycle making, it will be desirable to give some additional formula relating to the moment of inertia and the modulus of a tubular section.

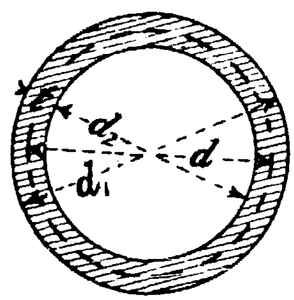


FIG. 94.

Let d_1, d, and d_2 (fig. 94) be the outside, mean, and inside diameters respectively, t the thickness, and A the area of the transverse section of the tube. From (12) for this section

$$I = \frac{\pi}{64}(d_1^4 - d_2^4) = \frac{\pi}{64}(d_1 - d_2)(d_1 + d_2)(d_1^2 + d_2^2) \quad . \quad (26)$$

Now, $d_1 - d_2 = 2\,t$, $d_1 + d_2 = 2\,d$, $d_1 = d + t$, $d_2 = d - t$.

Therefore,

$$(d_1^2 + d_2^2) = (d + t)^2 + (d - t^2) = 2(d^2 + t^2).$$

Substituting in (26)

$$I = \frac{\pi}{64} \,.\, 2\,t \,.\, 2\,d \,.\, 2(d^2 + t^2)$$

But $\pi\,d\,t = A$, therefore,

$$I = \frac{A}{16}(d_1^2 + d_2^2) = \frac{A}{8}(d^2 + t^2) \,. \quad . \quad . \quad . \quad (27)$$

Now,

$$Z = \frac{I}{\frac{d_1}{2}} = \frac{A}{8}\frac{(d_1^2 + d_2^2)}{d_1} = \frac{A}{8}\frac{(2\,d_1^2 - 4\,d_1\,t + 4\,t^2)}{d_1}$$

$$= \frac{A}{4}\left(d_1 - 2\,t + \frac{2\,t^2}{d_1}\right) = \frac{A}{4}\left(d_2 + \frac{2\,t^2}{d_1}\right) \quad . \quad . \quad . \quad . \quad (28)$$

If the tube be *thin*, t^2 will be small in comparison with d^2, and

$\frac{2\,t^2}{d_1}$ will be small in comparison with d_2. Equations (27) and (28) may then be written

$$I = \frac{\pi}{8}\, d^3\, t = \frac{A\, d^2}{8} \text{ approximately} \quad . \quad . \quad . \quad (29)$$

$$Z = \frac{\pi}{4}\, d^2\, t = \frac{A\, d_2}{4} \text{ approximately.} \quad . \quad . \quad . \quad (30)$$

The error introduced by using the approximate formula (30) for Z is on the safe side, and is very small for the ordinary tube sections used in cycle construction. Thus for a tube 1 inch diameter, 16 W.G., the exact value of Z is ·04140, that given by (30) is ·04102, the error being less than 1 per cent. in this case. If, however, d or d_1, be used instead of d_2 in formula (30) the error will be on the wrong side.

Table IV. gives the sectional areas, weights per foot run, and moduli of bending resistance for the ordinary sections of steel tubes used in cycle construction, the moduli having been calculated from the exact formula (28).

From (30) the transverse strength of a tube is proportional to its sectional area and to its internal diameter. If the internal diameter be kept constant, the transverse strength is proportional to the thickness. If the sectional area be kept constant, the transverse strength is proportional to the internal diameter. If the thickness be kept constant the strength is approximately proportional to the square of the diameter.

97. **Oval Tubes.**—We have already seen that the moment of inertia of an ellipse with major and minor axes h and b respectively is $\frac{\pi}{64}\, b\, h^3$.

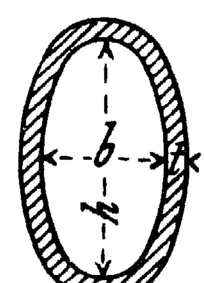


FIG. 95.

Let a second ellipse (fig. 95) be drawn outside the first and concentric with it, having its semi-axes the length t greater. The axes of the second ellipse will be $b + 2\,t$ and $h + 2\,t$ respectively, and its moment of inertia will be

$$\frac{\pi}{64}(b + 2\,t)\,(h + 2\,t)^3 =$$

$$\frac{\pi}{64}\Big\{ b\,h^3 + (2\,h^3 + 6\,b\,h^2)\,t + (12\,h^2 + 12\,b\,h)\,t^2$$
$$+ (24\,h + 8\,b)\,t + 16\,t^4 \Big\}$$

TAB.

SECTIONAL AREAS, WEIGHTS PER FOOT RUN, A

Outside diameter of tube		3″/8			1″/2			5″/8		
Imperial standard wire gauge	Thickness. Inches	W lbs. per foot length	A sq. in.	Z in³	W lbs. per foot length	A sq. in.	Z in.³	W lbs. per foot run	A sq. in.	Z in
No. 10	·128	·34	·0993	·0051	·52	·1496	·0116	·69	·1998	·0
11	·116	·33	·0944	·0051	·48	·1399	·0113	·64	·1855	·0
12	·104	·31	·0885	·0049	·45	·1294	·0108	·59	·1702	·0
13	·092	·28	·0818	·0048	·41	·1180	·0103	·53	·1541	·0
14	·080	·26	·0742	·0046	·36	·1056	·0096	·47	·1370	·0
15	·072	·24	·0685	·0044	·33	·0968	·0091	·43	·1251	·0
16	·064	·22	·0625	·0042	·30	·0877	·0085	·39	·1128	·0
17	·056	·19	·0561	·0039	·27	·0781	·0078	·35	·1001	·0
18	·048	·17	·0493	·0036	·24	·0682	·0070	·30	·0870	·0
19	·040	·15	·0421	·0032	·20	·0578	·0062	·25	·0736	·0
20	·036	·13	·0383	·0030	·18	·0525	·0057	·23	·0666	·0
21	·032	·12	·0345	·0027	·16	·0470	·0052	·21	·0596	·0
22	·028	·11	·0305	·0025	·14	·0415	·0046	·18	·0525	·0
23	·024	·09	·0265	·0022	·12	·0359	·0041	·16	·0453	·0
24	·022	·08	·0244	·0020	·11	·0330	·0038	·14	·0417	·0
25	·020	·08	·0223	·0019	·10	·0302	·0035	·13	·0380	·0
26	·018	·07	·0202	·0017	·09	·0273	·0032	·12	·0343	·0
28	·0148	·06	·0167	·0015	·08	·0226	·0027	·10	·0284	·0
30	·0124	·05	·0141	·0012	·07	·0190	·0023	·08	·0239	·0
32	·0108	·04	·0124	·0011	·06	·0166	·0020	·07	·0208	·0

Outside diameter of tube		1 1″/4			1 3″/8			1 1″/2		
No. 10	·128	1·56	·4511	·1151	1·73	·5013	·1433	1·91	·5516	·1
11	·116	1·43	·4132	·1074	1·59	·4587	·1334	1·74	·5043	·1
12	·104	1·29	·3744	·0992	1·43	·4152	·1228	1·58	·4560	·1
13	·092	1·16	·3347	·0903	1·28	·3708	·1115	1·41	·4069	·1
14	·080	1·02	·2940	·0809	1·12	·3255	·0996	1·23	·3569	·1
15	·072	·92	·2664	·0742	1·02	·2947	·0912	1·12	·3230	·1
16	·064	·82	·2384	·0673	·91	·2635	·0825	1·00	·2887	·0
17	·056	·73	·2101	·0600	·80	·2320	·0735	·88	·2540	·0
18	·048	·63	·1813	·0525	·69	·2001	·0642	·76	·2190	·0
19	·040	·53	·1521	·0446	·58	·1679	·0544	·63	·1836	·0
20	·036	·47	·1373	·0405	·52	·1514	·0494	·57	·1656	·0
21	·032	·42	·1224	·0364	·47	·1350	·0443	·51	·1476	·0
22	·028	·37	·1075	·0321	·41	·1185	·0391	·45	·1295	·0
23	·024	·32	·0924	·0278	·35	·1019	·0338	·38	·1113	·0
24	·022	·29	·0849	·0256	·32	·0935	·0311	·35	·1022	·0

V

Moduli of Bending Resistance of Steel Tubes

$\frac{3}{4}''$			$\frac{7}{8}''$			$1''$			$1\frac{1}{8}''$		
W lbs. per foot run	A sq. in.	Z in.³	W lbs. per foot run	A sq. in.	Z in.³	W lbs. per foot run	A sq. in.	Z in.³	W lbs. per foot run	A sq. in.	Z in.³
·86	·2501	·0336	1·04	·3003	·0493	1·21	·3506	·0681	1·38	·4009	·0900
·80	·2310	·0320	·96	·2766	·0466	1·11	·3222	·0640	1·27	·3677	·0843
·73	·2111	·0301	·87	·2519	·0436	1·01	·2927	·0595	1·15	·3335	·0781
·60	·1902	·0280	·78	·2263	·0402	·91	·2624	·0546	1·03	·2981	·0714
·58	·1684	·0256	·69	·1998	·0365	·80	·2312	·0493	·91	·2626	·0641
·53	·1534	·0238	·63	·1816	·0337	·73	·2099	·0455	·82	·2382	·0589
·48	·1379	·0218	·56	·1631	·0308	·65	·1882	·0414	·74	·2133	·0535
·42	·1221	·0197	·50	·1441	·0277	·57	·1661	·0371	·65	·1881	·0479
·37	·1059	·0175	·43	·1247	·0244	·50	·1436	·0326	·56	·1624	·0419
·31	·0893	·0150	·36	·1050	·0210	·42	·1207	·0279	·47	·1364	·0357
·28	·0807	·0137	·33	·0949	·0191	·38	·1090	·0253	·43	·1232	·0325
·25	·0722	·0124	·29	·0847	·0172	·34	·0973	·0228	·38	·1039	·0292
22	·0635	·0110	·26	·0745	·0152	·30	·0855	·0202	·33	·0965	·0258
·19	·0547	·0096	·22	·0642	·0133	·25	·0736	·0175	·29	·0830	·0223
·17	·0503	·0089	·20	·0590	·0122	·23	·0676	·0161	·26	·0762	·0206
·16	·0459	·0082	·19	·0537	·0112	·21	·0616	·0148	·24	·0694	·0188
·14	·0414	·0074	·17	·0485	·0102	·19	·0555	·0134	·22	·0626	·0170
·12	·0342	·0062	·14	·0400	·0085	·16	·0458	·0111	·18	·0516	·0140
·10	·0287	·0052	·12	·0336	·0071	·13	·0385	·0094	·15	·0433	·0119
.09	·0251	·0046	·10	·0293	·0063	·12	·0336	·0082	·13	·0378	·0104

$1\frac{5}{8}''$			$1\frac{3}{4}''$			$1\frac{7}{8}''$			$2''$		
2·08	·6019	·2090	2·25	·6521	·2467	2·43	·7024	·2874	2·60	·7526	·3312
1·90	·5499	·1938	2·06	·5954	·2283	2·21	·6409	·2657	2·37	·6865	·3058
1·72	·4969	·1777	1·86	·5377	·2090	2·00	·5785	·2428	2·14	·6194	·2792
1·53	·4431	·1608	1·66	·4792	·1888	1·78	·5153	·2191	1·91	·5515	·2516
1·34	·3883	·1430	1·45	·4197	·1676	1·56	·4511	·1942	1·67	·4826	·2228
1·21	·3513	·1306	1·31	·3795	·1530	1·41	·4078	·1771	1·51	·4361	·2029
1·08	·3138	·1179	1·17	·3390	·1379	1·26	·3641	·1594	1·34	·3892	·1825
·95	·2760	·1047	1·03	·2980	·1223	1·11	·3200	·1413	1·18	·3420	·1617
·82	·2378	·0911	·89	·2566	·1063	·95	·2755	·1227	1·02	·2944	·1402
·69	·1992	·0770	·74	·2150	·0898	·80	2307	·1036	·85	·2465	·1184
·62	·1797	·0698	·67	·1938	·0814	·72	·2080	·0938	·77	·2221	·1071
·55	·1601	·0625	·60	·1727	·0728	·64	·1853	·0839	·68	·1978	·0958
·49	·1405	·0551	·53	·1515	·0642	·56	·1625	·0739	·60	·1735	·0843
·42	·1207	·0476	·45	·1301	·0554	·48	·1396	·0638	·51	·1490	·0727
·38	·1108	·0438	·41	·1194	·0510	·44	·1281	·0586	·47	·1367	·0669

Therefore the moment of inertia of the area included between the ellipses is

$$I = \frac{\pi}{64}\left\{(2h^3 + 6bh^2)t + (12h^2 + 12bh)t^2 + (24h + 8b)t + 16t^4\right\} \quad . \quad . \quad . \quad (31)$$

If t is small in comparison with b and h, the second, third, and fourth terms in the expression for I are smaller and smaller compared with the first, and may be neglected. Therefore, the moment of inertia of the figure is approximately

$$I = \frac{\pi}{32} h^2 t (h + 3b) \quad . \quad . \quad . \quad (32)$$

The modulus of bending resistance is approximately

$$Z = \frac{\pi}{16} h t (h + 3b) \quad . \quad . \quad . \quad (33)$$

When a tube of circular section is flattened to form an oval tube, its thickness will be nearly uniform throughout, but in the oval tube section, above discussed, the thickness is not constant throughout, but is a little less than t except at the ends of the major and minor axes. The strength of an oval tube of uniform thickness will therefore be under-estimated if the formula (33) be used, so that the error is on the safe side.

The area of the ellipse is $\frac{\pi}{4} a b$; and in the same way as above it can be shown that the area included between the two ellipses is

$$A = \frac{\pi}{2}(b + h)t \quad . \quad . \quad . \quad (34)$$

Therefore, (32) and (33) may be respectively written,

$$I = \frac{(3b + h)}{16(b + h)} A h^2 \quad . \quad . \quad . \quad (35)$$

$$Z = \frac{(3b + h)}{8(b + h)} A h \quad . \quad . \quad . \quad (36)$$

An oval tube of uniform thickness will be stronger than indicated by formula (36). This is clearly shown by figure 96, which represents a quarter section; the modulus given in (36) is that of

the tube whose inner surface is represented by the dotted line. The inner continuous line represents a tube of the same sectional area A, and of uniform thickness. It has the area b in excess of the dotted tube, and the areas a and c deficient, $a + c = b$. It is evident from the figure that the moment of inertia of b about the minor axis is greater than that of a and c; and therefore the tube of uniform thickness is slightly stronger than the section above discussed.

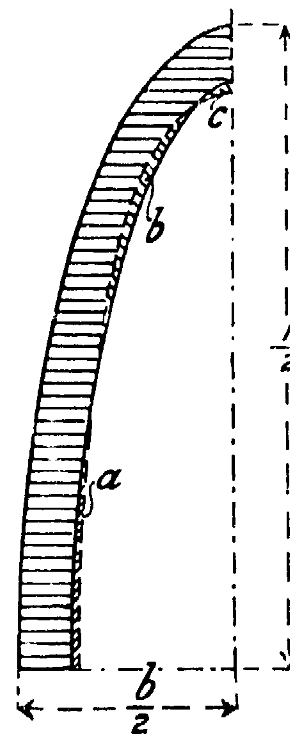


FIG. 96.

98. **D Tubes.**—Tubes of **D** section have been recently introduced for the lower back fork of a bicycle; it will be instructive to investigate their bending resistances here. We will assume that the outline of the **D** tube is made up of a semicircle and its diameter (fig. 97). Let r be the radius and h the diameter of the semicircle, t the thickness, which we will consider very small in comparison with r, and A the sectional area of the **D** tube. First, consider the moment of inertia about the axis $a\ a$ at right angles to the flat side of the tube. The moment of inertia of the rectangle of depth h and width t is $\frac{1}{12} h^3 t$, the moment of inertia of the semicircle is $\frac{\pi}{16} h^3 t$, therefore the moment of inertia of the **D** tube about the axis $a\ a$ is

$$I = \left(\frac{1}{12} + \frac{\pi}{16}\right) h^3 t \quad . \quad . \quad . \quad (37)$$

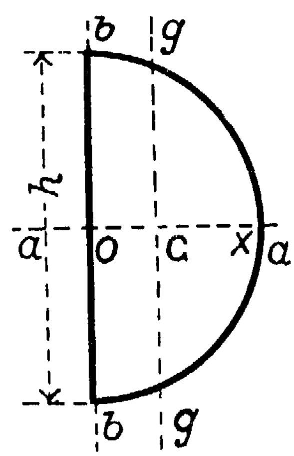


FIG. 97.

The modulus of the section, about the same axis, is

$$Z = \left(\frac{1}{6} + \frac{\pi}{8}\right) h^2 t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Consider now the moment of inertia about the axis $b\ b$, coinciding with the flat side of the **D** tube. The moment of inertia of

the flat side is $\frac{1}{12} h t^3$, that of the semicircle is $\frac{\pi}{16} h^3 t$; therefore the moment of inertia of the section of the **D** tube is

$$\frac{1}{12} h t^3 + \frac{\pi}{16} h^3 t.$$

If t be small in comparison with h, the first term in this expression may be neglected in comparison with the second, and therefore,

$$I = \frac{\pi}{16} h^3 t, = \frac{\pi}{2} r^3 t, \text{ approximately} \quad . \quad . \quad . \quad (39)$$

But the I just found is not about an axis through the centre of figure; this we now proceed to find. Let G be the centre of figure; the distance OG can be found as follows: The moment of the semicircle about the axis bb is $2r^2 t$ (see sec. 50), that of the straight side about the same axis is zero, the total moment of the **D** tube about the axis bb is therefore $2r^2 t$. But the total moment is also equal to the total area multiplied by the distance OG; therefore $2r^2 t = (2r + \pi r) t \times OG$,

$$\text{And } OG = \frac{2r^2}{(2+\pi) r} = \frac{2}{2+\pi} r = .389 r. \quad . \quad . \quad . \quad (40)$$

Let I_0 be the moment of inertia about an axis gg passing through G parallel to bb; then by section 92

$$I = I_0 + (2+\pi) r t \,.\, \frac{4}{(2+\pi)^2} r^2$$

$$\text{Therefore, } I_0 = \frac{\pi}{2} r^3 t - \frac{4 r^3 t}{(2+\pi)} = \left(\frac{\pi}{2} - \frac{4}{2+\pi}\right) r^3 t \quad . \quad . \quad (41)$$

But $A = (2+\pi) r t$; therefore we may write

$$I_0 = \left\{\frac{\pi}{2(2+\pi)} - \frac{4}{(2+\pi)^2}\right\} A r^2 \quad . \quad . \quad . \quad . \quad (42)$$

Z, the modulus of bending resistance about the axis gg is equal to $\frac{I_0}{GX}$, X being the extremity of the radius through OG. Now,

$$GX = OX - OG = r - \frac{2}{2+\pi} r = \frac{\pi}{2+\pi} r.$$

$$\therefore Z = \left\{\frac{1}{2} - \frac{4}{\pi(2+\pi)}\right\} A r = .2524 A r \quad . \quad (43)$$

Let d be the diameter of a round tube equal in perimeter to the **D** tube. Then $\pi\, d = (2 + \pi) r$,

$$\therefore r = \frac{\pi}{2 + \pi}\, d = \cdot 6110\, d.$$

Substituting this value of r in (43), we get,

$$Z = \left\{ \frac{\pi}{2\,(2 + \pi)} - \frac{4}{(2 + \pi)^2} \right\} A\, d = \cdot 1542\, A\, d \quad (44)$$

The Z of the original round tube is approximately $\frac{1}{4}\, A\, d$, so the strengths of a round tube and the **D** tube into which it can be pressed are in the ratio of ·2500 to ·1542, *i.e.* 1000 to 617.

But since a **D** tube is used when the space $O\,X$ is limited, it would seem fairer to compare it with a round tube of equal weight and of diameter $O\,X$. The Z of a round tube of diameter $O\,X$ is $\cdot 25\, A\, r$. Comparing this value with that in (43), it is seen that the strength of the **D** tube is slightly greater than that of a round tube of equal weight, and of diameter equal to the smallest diameter of the **D** tube, the ratio being ·252 to ·2500, *a difference of less than one per cent. in favour of the* **D** *tube.*

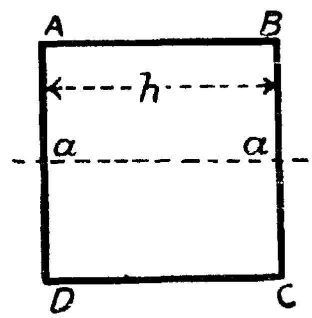


FIG. 98.

99. **Square and Rectangular Tubes.**—Consider the I of a square tube of section $A\,B\,C\,D$ (fig. 98), about an axis $a\,a$ parallel to the side $A\,B$. The I of each of the sides $B\,C$ and $D\,A$ is $\frac{h^3\, t}{12}$, that of each of the sides $A\,B$ and $C\,D$ is $h\, t\,.\,\frac{h^2}{4}$; therefore, for the whole section

$$I = 2\,.\,\frac{h^3\, t}{12} + 2\,.\,h\, t\,.\,\frac{h^2}{4} = \frac{2}{3}\, h^3\, t \quad (45)$$

The total sectional area is $4\, h\, t$, therefore

$$I = \frac{1}{6}\, A\, h^2 \quad (46)$$

also, $$Z = \frac{I}{\frac{h}{2}} = \frac{1}{3}\, A\, h \quad (47)$$

Let d be the diameter of a round tube of the same perimeter as the square tube ; then $4\,h = \pi\,d$

$$\therefore\ h = \frac{\pi}{4}\,d = \cdot 7854\,d, \text{ and}$$

$$Z = \frac{\pi}{12}\,A\,d = \cdot 2618\,A\,d \quad . \quad . \quad . \quad . \quad . \quad . \qquad (48)$$

hence, comparing with (30), the moduli of bending resistance of the square tube and of the original round tube are in the ratio of $\frac{\pi}{12}$ to $\frac{1}{4}$, or of π to 3, *i.e.* 1047 to 1000, in favour of the square tube. Compared with a round tube of equal sectional area, but of the same diameter as the side of the square tube, the ratio is $\frac{A\,d}{3}$ to $\frac{A\,d}{4}$, *i.e.* $\frac{1}{3}$ to $\frac{1}{4}$, or 133·3 to 100 ; *i.e. the square tube is* 33·3 *per cent. stronger than the round tube of equal area and diameter.*

Rectangular Tubes.—If a round tube be drawn into a rectangular tube of the same thickness, perimeter, and sectional area, it can be shown that the strength of the latter will be greatest when its depth h is three times its width b.

For any rectangular section, approximately

$$I = 2\,b\,t\left(\frac{h}{2}\right)^2 + 2\,\frac{t\,h^3}{12} = \frac{t\,h^2}{2}\left(b + \frac{h}{3}\right) \quad . \quad . \qquad (49)$$

$$Z = t\,h\left(b + \frac{h}{3}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (50)$$

For the strongest rectangular tube, (49) becomes

$$I = 9\,t\,b^3 = \frac{1}{8}\,A\,h^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (51)$$

and,
$$Z = \frac{1}{4}\,A\,h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (52)$$

Comparing (33) and (50), it is seen that a thin rectangular tube is stronger than an elliptical tube of the same depth, width, and thickness in the ratio $16 : 3\,\pi$. Now the ratio of the perimeters, and therefore the weights, is never greater than $4 : \pi$; this being the value when the ellipse and rectangle become a circle

and square respectively. Weight for weight, then, the rectangular has at least $\frac{4}{3}$ times the strength of the elliptical tube.

That the rectangular is stronger than the elliptical tube of equal depth, width, and sectional area, can be easily shown from first principles, as follows: Figure 99 shows quadrants of rectangular and elliptical tubes of equal sectional area. Since the perimeter of the ellipse is less than that of the rectangle, its thickness is greater. Let a portion a of the ellipse be marked off equal in width to the corresponding part of the rectangle, so that the moments of inertia about the axis OX are equal. The part b is common to both ellipse and rectangle, and there remain only the parts c. That belonging to the rectangle is at a much greater distance from the axis OX than that belonging to the ellipse; its moment of inertia is therefore greater, and the rectangular is stronger than the elliptical tube to resist bending.

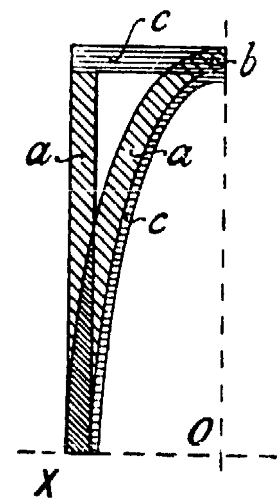


FIG. 99.

CHAPTER XII

SHEARING, TORSION, AND COMPOUND STRAINING ACTION

100. **Compression.**—The laws relating to simple compressive stress are exactly the same as those of simple tension, the formula (1), (2), (3), and (4), of chapter x. will apply, p being in this case the compressive stress, e the compression per unit of length, and E the modulus of elasticity for compression. For a homogeneous material with perfect elasticity, as above defined, E would be the same for tension and compression.

On a bar which is short in comparison to its diameter, if the compressive stress be increased above the elastic limit of compression, the bar gives way ultimately by lateral yielding. If the material be hard, the bar may actually split up into several pieces. If of a soft, ductile material it will bulge gradually in the middle while being shortened in length.

101. **Compression or Tension combined with Bending.**—If a bar be simultaneously subjected to bending, and a pull or thrust parallel to its axis, the maximum stress on the section is the sum of the separate stresses due to the separate straining actions. If the bar be subjected to a pull P, and a bending-moment M, A being the area and Z the modulus of the section, the maximum tensile stress is

$$f = \frac{P}{A} + \frac{M}{Z} \qquad (1)$$

and the minimum tensile stress is

$$f^1 = \frac{P}{A} - \frac{M}{Z} \qquad (2)$$

For circular tubes of small thickness, substituting the value of Z from (30), section 96,

$$f = \frac{P}{A} + \frac{4\,M}{A\,d} \qquad (3)$$

The bending-moment may be produced by applying the pull P at a distance x from the neutral axis of the section (fig. 100). In this case $M = Px$ and (3) may be written

$$f = \frac{P}{A} + \frac{4Px}{Ad} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the bar be subjected to a compression P and a bending-moment M, equations (1), (3), and (4) give the maximum *compressive* stress on the section, equation (2) the minimum compressive stress.

102. **Columns.**—If a long bar be subjected to tension, any slight deviation from straightness (fig. 100) will, under the action of the forces, tend to get less. If, on the other hand, the bar be subjected to compression, the deviation from straightness will tend to get greater, and the bar will give way by bending (fig. 101).

The stresses on a straight short column supporting a load, placed eccentrically, are given by formulæ (1) and (2).

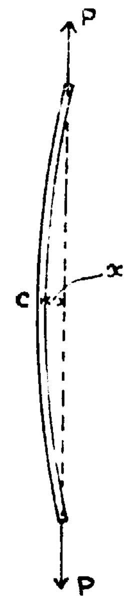


FIG. 100.

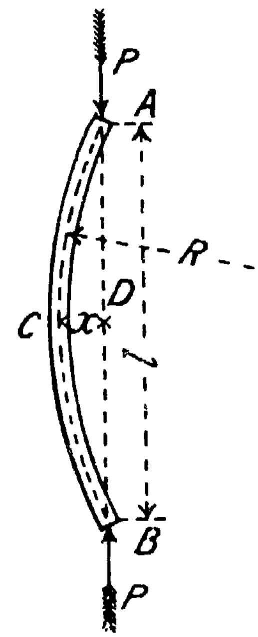


FIG. 101.

Example.—A bicycle tube 1 in. diameter, 16 W.G., is subjected to a compressive force, the axis of which is $\frac{1}{4}$ in. from the axis of the tube. Find the breaking load, the breaking stress of the material being 30 tons per sq. in. From Table IV., $A = .1882$ sq. in., $Z = \cdot 0414$ in.^{3}, also $M = \frac{1}{4}P$ inch-lbs. $f = 30 \times 2240$ lbs. per sq. in.; substituting in (1)

$$30 \times 2240 = \frac{P}{\cdot 1882} + \frac{P}{4 \times \cdot 0414},$$

from which, $P = 5921$ lbs.

If the load were placed exactly co-axial with the tube, it would reach the value given by,

$$\frac{P}{\cdot 1882} = 30 \times 2240$$

i.e., $P = 12650$ lbs.

103. **Limiting Load on Long Columns.**—If, under the action of the load, the deviation x becomes greater, the bending-moment also becomes greater without any addition being made to the load; thus the deviation once started, may rapidly increase until fracture of the column takes place.

Let the section of the column be such that, under the action of the load, its neutral axis bends into a circular arc $A\,C\,B$ (fig. 101) of radius R. Let $A\,D\,B$ be the chord, $C\,D$ the greatest deviation, and $C\,E$ a diameter of the circle. Then, by the well-known proposition in elementary geometry,

$$C\,D \times D\,E = A\,D \times D\,B.$$

i.e. neglecting the difference between $C\,E$ and $D\,E$,

$$2\,R\,x = \frac{l^2}{4} \text{ approximately.}$$

But $R = \dfrac{E\,I}{M}$, from (17), chap. xi., and $M = P\,x$. Substituting,

$$P = \frac{8\,E\,I}{l^2} \quad \ldots\ldots\ldots\ldots \quad (5)$$

If the load be less than that given by (5), no deviation will take place.

If the column be of constant section throughout its length its neutral axis bends into a curve of sines, and it can be shown that the limiting load is

$$P = \frac{\pi^2\,E\,I}{l^2} \quad \ldots\ldots\ldots\ldots \quad (6)$$

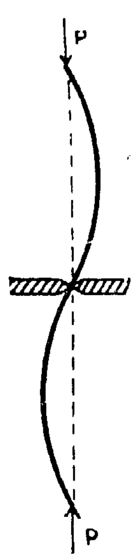


FIG. 102.

If the middle section of the column be prevented from deviating laterally, it will bend into the form shown in figure 102. In this case the length of the segment of the curve corresponding to figure 101 is half the total length, and the corresponding load will be

$$P = \frac{4\,\pi^2\,E\,I}{l^2} \quad \ldots\ldots\ldots\ldots \quad (7)$$

Again, if the ends of the column be held in such a manner that the directions of the axis at the end are always the same, it will give way by bending as shown in figure 103. The segment

$b\,d$ in this case is of the same shape as the curve in figure 101, while the portions $a\,b$ and $e\,d$ are of the same form as $b\,c$ and $d\,c$. In this case, therefore, the length of the segment $b\,d$ is $\frac{l}{2}$, and the corresponding limiting load is given by the formula

$$P = \frac{4\,\pi^2 E\,I}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If the column be fixed at one end (fig. 104), held laterally but free to turn at the other,

$$P = \frac{2\,\pi^2 E\,I}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

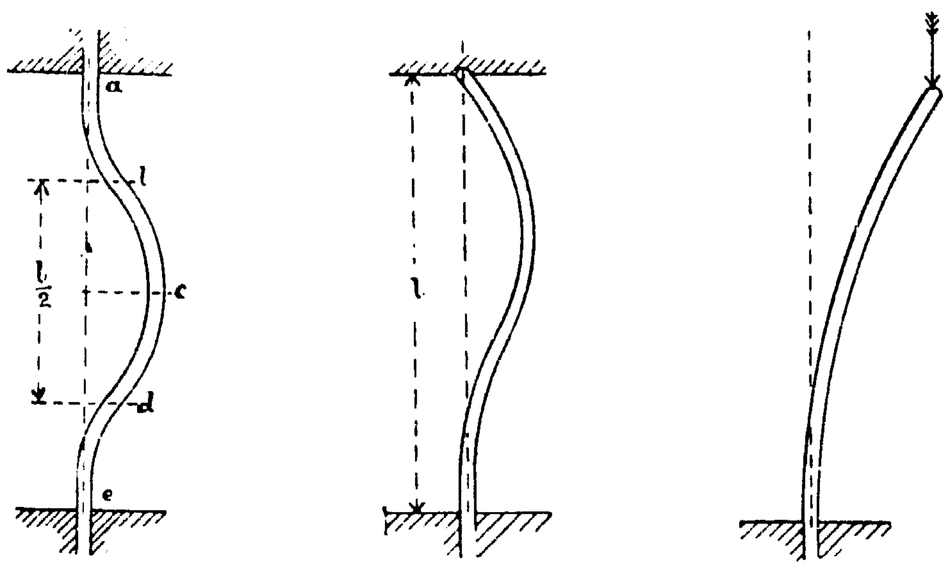


FIG. 103. FIG. 104. FIG. 105.

If the column be fixed at one end and quite free at the other end (fig. 105),

$$W = \frac{\pi^2 E\,I}{4\,l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

These are known as Euler's formulæ, and are only applicable to bars or columns in which the length l is great as compared with the least transverse dimension. l is the length before bending; though in the figures, in which the bending is greatly exaggerated, it is marked as *after* bending.

104. **Gordon's Formula for Columns.**—The pieces of tube used in bicycle building are too long to have the simple compression formula applied to them, and too short for the application

of Euler's formula. A great many experiments on columns, principally cast iron, have been made by Hodgkinson, and Gordon has suggested an empirical formula which agrees very closely with his experiments. For thin tubes, Gordon's formula becomes

$$\frac{W}{A} = \frac{f}{1 + \dfrac{8\,l^2}{c\,d^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

f and c being constants depending on the material.

Actual experiments on the compressive strengths of weldless steel tubes are wanting, but taking $f = 30$ tons per sq. in., and $c = 32{,}000$, Gordon's formula becomes

$$\frac{W}{A} = \frac{67200}{1 + \dfrac{l^2}{32000\,d^2}} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Example.—A tube is 1 in. diameter, No. 16 W.G., 20 in. long; required the crushing load by Gordon's formula.

From Table IV., page 113, $A = \cdot 1882$;

$$\therefore \frac{W}{\cdot 1882} = \frac{67200}{1 + \dfrac{400}{32000 \times 1}} = \frac{67200}{1{\cdot}0125}$$

from which,

$$W = 12490 \text{ lbs.},$$

slightly less than for a short length of the same tube (sec. 102).

105. **Shearing.**—Let $A\,B\,C\,D$ (fig. 106) be a small square prism of unit width perpendicular to the paper, subjected to shearing stress on the planes $A\,B$ and $C\,D$. If the planes $A\,B$ and $C\,D$ be very close to each other, the shearing stress will be the same on both. If q be the shearing stress per unit of area, the downward force acting at $A\,B$ and the upward force at $C\,D$ will each be $q \times A\,B$. But since the portion $A\,B\,C\,D$ is at rest, the couple formed by the forces at $A\,B$ and $C\,D$ must be

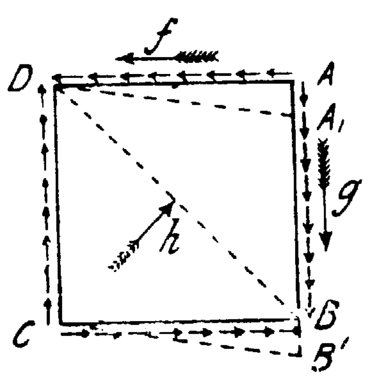


FIG. 106.

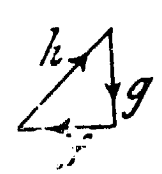


FIG. 107.

balanced by an equal and opposite couple, formed by forces acting at $A\ D$ and $B\ C$, since no force acts normally at the surfaces $A\ B$ and $C\ D$. Thus the shearing stress on the sides $A\ D$ and $B\ C$ is equal to that on $A\ B$ and $D\ C$; or the shearing stress on â plane is always accompanied by an equal shearing stress on a plane at right angles to the former, and to the direction of the shearing stress on the former plane.

Transverse Elasticity.—Under the action of the shearing forces the square $A\ B\ C\ D$ (fig. 106) will be distorted into a rhombus, $A^1\ B^1\ C\ D$, the angle of distortion $A\ D\ A^1$ being proportional to the shearing stress. Let ϕ be this angle and q the shearing stress producing it; then

$$q = C\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

C being the *modulus of transverse elasticity*, or the *coefficient of rigidity* of the material.

Shearing Stress equivalent to Simultaneous Tension and Compressive Stresses.—Draw a diagonal $B\ D$ (fig. 106); the triangular prism $A\ D\ B$ is in equilibrium under the action of the three forces, f, g, and h, acting on its sides, which can therefore be represented by the sides of a triangle (fig. 107). f and g being equal, the force h is evidently at right angles to the side $B\ D$. The triangles $A\ B\ D$ and $f\ g\ h$ are similar; that is, the forces f, g and h are proportional to the lengths of the sides on which they act; the stress per unit area must therefore be the same for the three sides $A\ B$, $B\ D$, and $D\ A$. Thus, the stress on the plane $B\ D$ is a compressive stress of the same intensity as the shearing stress on the planes $A\ B$ and $A\ D$.

In the same way it may be shown that a *tensile* stress of equal magnitude exists on the plane $A\ C$. Thus, in any body a pair of shearing stresses on two planes at right angles are equivalent to a pair of compressive and tensile stresses respectively on two planes mutually at right angles, and inclined 45° to the former planes.

106. **Torsion.**—If a long bar be subjected to two equal and opposite couples acting at its ends, the axes of the couples being parallel to the axis of the bar (fig. 108), it is said to be subjected to *torsion*. The moment of the couple applied is called the *twisting-moment* on the bar. If one end be rigidly fixed, the

other end will, under the action of the twisting-moment, be displaced through a small angle, and any straight line on the surface of the bar originally parallel to the axis will be twisted into a spiral curve $a\ a$. If the twisting-moment be increased indefinitely, the bar will ultimately break, the total angle of twist before breaking depending on the nature of the material.

Let figure 109 be the longitudinal elevation of a thin tube of mean radius r and thickness t, subjected to a twisting-moment

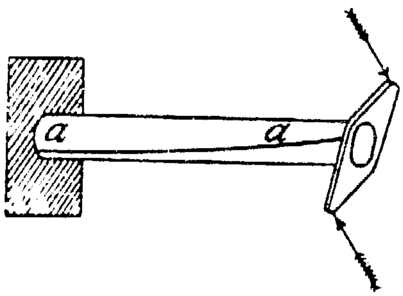


FIG. 108.

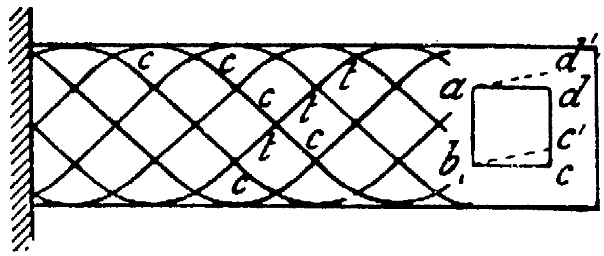


FIG. 109.

T foot-lbs. A square, $a\ b\ c\ d$, drawn on the surface of the tube becomes distorted while in a strained condition into the rhombus $a\ b\ c'\ d'$. Thus, every transverse section of the tube is subjected to a shearing stress. If the tube be of uniform diameter and thickness, this shearing stress, q, will be the same throughout, provided the thickness is very small in comparison with the diameter.

The sectional area of the tube is $2\pi t r$; and since q is the shear on unit area, the total shear on the section is $2\pi q t r$. The shearing-force on each element of the section acts at a distance r from the centre of the tube; the moment of the total shearing-force is therefore $2\pi q t r^2$. This must be equal to the twisting-moment T_1 applied to the end; therefore

$$T_1 = 2\pi q t r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Thus the twisting-moment which can be transmitted by a thin tube of circular section is proportional to the square of its radius or diameter and to its thickness.

107. **Torsion of a Solid Bar.**—In a solid cylinder of radius r_1, imagine the square $a\ b\ c\ d$ (fig. 109) drawn on a concentric cylindrical surface of radius r; it is easily seen that the angle of distortion of the fibres, ϕ, or $d\ a\ d^1$, is proportional to r. If ϕ_1 be

the angle of distortion for a square drawn on the surface of the cylindrical rod, q and q_1 the shearing stresses at radii r and r_1 respectively, then evidently

$$\phi = \frac{\phi_1}{r_1} r;$$

and therefore

$$q = \frac{q_1}{r_1} r.$$

If now the solid rod be considered to be divided into a number of thin concentric tubes, all of the same thickness, t, the area of the tube of radius r is $2 \pi t r$, the total shear on this tube is

$$\frac{2 \pi q_1}{r_1} t r^2,$$

and the twisting-moment resisted is

$$\frac{2 \pi q_1}{r_1} t r^3.$$

The sum of the moments of all the concentric tubes into which the rod is divided is easily found, by one of the simplest examples in the integral calculus, to be

$$T = \frac{\pi q_1}{2} r_1^3,$$

or,

$$T = \frac{\pi}{16} d^3 q_1 = \frac{d^3 q}{5} \text{ approx.} \quad . \quad (15)$$

108. **Torsion of Thick Tubes.**—If r_1 and r_2 be the external and internal radii of a hollow tube, the sum of the twisting-moments (45) of the very thin concentric tubes into which it may be divided —and, therefore, the twisting-moment such a tube can resist—is

$$\frac{\pi (r_1^4 - r_2^4) q_1}{2 r_1}$$

or

$$T = \frac{\pi}{16} \frac{(d_1^4 - d_2^4) q_1}{d_1} \quad . \quad . \quad . \quad . \quad (16)$$

The quantity $\frac{\pi}{16} \frac{(d_1^4 - d_2^4)}{d_1}$ depends simply on the dimensions

of the section of the tube, and may be called the *modulus of resistance to torsion*; it may be denoted by the symbol Z_T. Then

$$T = Z_T \cdot q_1.$$

Comparing Z_T with Z, chapter xi., it will be seen that the modulus of resistance of a circular tube or solid bar to torsion is twice its modulus of resistance to bending. The strength of any tube to resist bending can therefore be obtained by multiplying the modulus from Table IV., page 112, by twice the maximum shear q_1.

109. **Lines of Direct Tension and Compression on a Bar subject to Torsion.**—From what has been said in section 105, there will be a compressive stress on the plane $a\ c$, and a tensile stress on the plane $b\ d$ (fig. 109). This holds for every point on the surface of the tube. Now if the tube be split up into a number of narrow strips by the spiral lines $t\ t$, inclined 45° to the axis (fig. 10), the tensile stresses can be transmitted just as before. The spiral lines $t\ t$ are said to be *tension lines*, and the spiral lines $c\ c$ at right angles *compression lines*. If the twisting-moment be in the opposite direction, however, the pressure and tension spiral lines will be interchanged, and the split tube will not be able to transmit the twisting-moment.

110. **Compound Stress.**—If the straining actions on any part of a structure be all parallel to one plane, the stress on any plane section, at right angles to the plane of the straining actions, can be resolved into a normal stress, tension or compression—and a tangential stress, shearing. It can be shown that any system of stress in two dimensions is equivalent to a pair of normal stresses on two planes mutually at right angles, and that the stress on one of these planes is greater than, that on the other plane less than, on any other plane section of the material. On any other plane the stress will have a tangential component.

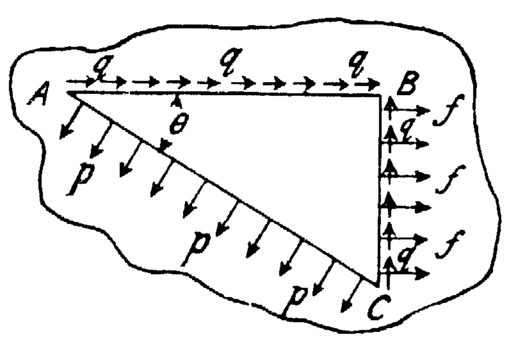


FIG. 110.

An important case of compound stress is that of a shaft subjected to bending and torsion; a section at right angles to the

axis of the shaft is subjected to a normal stress, f, and simultaneously to a torsional shearing stress, q. Consider a small portion of a material (fig. 110) subjected to stresses parallel to the plane of the paper. Let $A\ B\ C$ be a small prism, of unit depth at right angles to the paper, the face $B\ C$ being subjected to a normal stress, f, and a tangential stress, q. From section 106 we know that an equal shearing stress, q, must exist on the face $A\ B$. Let us find the magnitude of the stress p on the face $A\ C$, on which the stress shall be wholly normal.

Considering the equilibrium of the prism $A\ B\ C$, and resolving the forces on the three faces parallel to the side $A\ B$, we have

$$p \,.\, A\ C \,.\, \sin\theta - q \,.\, A\ B - f \,.\, B\ C = 0$$

or

$$(p - f)\tan\theta = q \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Similarly resolving the forces parallel to $B\ C$, we get,

$$p \,.\, A\ C \,.\, \cos\theta - q \,.\, B\ C = 0$$

or

$$p = q\tan\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Multiplying (17) and (18) together, we get

$$p\,(p - f) = q^2$$

from which

$$p = \tfrac{1}{2}\{f \pm \sqrt{f^2 + 4\,q^2}\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

the two values of p in (19) are the maximum and minimum normal stresses on the material. That is, the tension f and the shear q, on the face $B\ C$, produce on some plane $A\ C$ the maximum tensile stress $\frac{1}{2}\{f + \sqrt{f^2 + 4\,q^2}\}$, and on another plane the minimum tensile stress $\frac{1}{2}\{f - \sqrt{f^2 + 4\,q^2}\}$; the latter plane being at right angles to the former.

If the stresses on two planes at right angles be wholly normal and of equal intensity, it can easily be shown that the stress on any other plane is wholly normal and of the same intensity. If the normal stress be compression, the whole system of stress is of the nature of fluid pressure. If there be a tensile stress on one plane and an equal compressive stress on the plane at right angles,

it has already been shown that this is equivalent to shearing stresses of the same intensity on two planes at angles of 45° with the planes of the normal stresses. This pair of shearing stresses tends to distort the body but not to alter its volume, whereas fluid pressure tends to alter the volume but not the shape of the body.

Any set of stresses in two dimensions can be expressed as the sum of a fluid stress and a shearing stress. Let two planes, A and B, at right angles be subjected to normal tensile stresses of intensity, p and q, respectively. Then this state of stress is equivalent to the sum of two states of stress, the first being a tensile stress $\frac{p+q}{2}$ on both planes A and B, the second a tensile stress $\frac{p-q}{2}$ on A and an equal compressive stress on the plane B. For $p = \frac{p+q}{2} + \frac{p-q}{2}$, and $q = \frac{p+q}{2} - \frac{p-q}{2}$. This principle will be made use of when discussing the outer cover of a pneumatic tyre.

111. **Bending and Twisting of a Shaft.**—In a circular shaft of diameter, d, subjected to a bending-moment, M, and a twisting-moment, T, the normal stress due to the bending-moment is

$$f = \frac{M}{\frac{\pi}{32} d^3},$$

and the shearing-stress due to the twisting-moment is

$$q = \frac{T}{\frac{\pi}{16} d^3}.$$

Substituting these values in (19),

$$p = \frac{1}{\frac{\pi}{16} d^3}\left\{ M + \sqrt{M^2 + T^2} \right\}.$$

if the shaft be subjected to a twisting-moment, T_e, which would produce the same stress, p,

$$p = \frac{T_e}{\frac{\pi}{16} d^3},$$

and T_e is said to be the twisting-moment equivalent to the given bending-moment and twisting-moment acting simultaneously. Comparing the two expressions for p, we get

$$T_e = M + \sqrt{M^2 + T^2} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

Similarly, the equivalent bending-moment is

$$M_e = \tfrac{1}{2}\, T_e = \tfrac{1}{2}\left\{ M + \sqrt{M^2 + T^2} \right\} \quad . \quad . \quad . \qquad (21)$$

CHAPTER XIII

STRENGTH OF MATERIALS.

112. **Stress, Breaking and Working.**—Each part of a machine must be capable of resisting the greatest straining actions that may come on it. This condition fixes, as a rule, the smallest possible section of the part below which it is not permissible to go. In ordinary machines, where mere mass is sometimes requisite, the section actually used may often with advantage be considerably greater than the minimum; but in bicycles, since 'lightness' is always sought after, though it should always be secondary to 'strength,' the actual section used must not be very much greater than the minimum consistent with safety. The magnitude of the stress on any piece depends on the general configuration of the machine and of the arrangement of the external forces acting on it. The strength of the various parts depends on the physical qualities of the materials of which they are made, as well as on their section; this we will now proceed to discuss.

Breaking Stress.—If a load be applied at the end of a bar and be gradually increased, the bar will ultimately break under it. If the bar be of unit section—one square inch—the load on it at the instant of breaking is called the *breaking tensile strength* of the material. A great number of experiments have been made from time to time on the strength of materials, and the values of the breaking tensile strength for all materials used in construction are fairly accurately known.

Factor of Safety.—One method of designing parts of a machine or structure is to fix arbitrarily on a *working stress* which shall not be exceeded. This working stress is got by dividing the breaking stress of the material, as determined by experiment, by an arbitrary

number called a *factor of safety*. This factor of safety varies with the nature of the material used, and with the conditions to which the structure is subjected. Professor Unwin, in 'Elements of Machine Design,' gives a table of factors of safety, the factor varying from 3 for wrought iron and steel supporting a dead load, to 30 for brickwork and masonry subjected to a varying load. The factor of safety should be large for a material that can be easily broken by impact, and may be low for a material that undergoes considerable deformation before fracture actually takes place.

113. **Elastic Limit.**—We have already seen (sec. 81) that the application of a load to a bar of what might be popularly called a rigid material produces an elongation, and that this elongation is proportional to the load applied up to a certain limit. If not loaded beyond this limit, on removing the load the bar returns to its original length, and no permanent alteration has been made. If, however, the load applied be greater than the above limit, the elongation produced by it becomes greater proportionally, and on the load being removed the bar is found to be permanently increased in length. The stress beyond which the elongation is no longer proportional to the load, is called the *elastic limit.*

Since the elongation is in most metals proportional to the load applied up to this point, it has also been called the proportional limit (German, 'Proportionalitätsgrenze'). In a few metals—cast iron, brass—there is no well-defined proportional limit.

The total elongation of a bar loaded up to a stress just inside the elastic limit is a very small fraction of its original length. On increasing the load beyond the elastic limit and up to the breaking point, the elongation before fracture occurs, in the case of most materials, is a very much greater fraction of the original length.

Table V. gives the breaking and elastic strengths and coefficients of elasticity of most of the materials used in cycle making ; the figures are taken from Professor Unwin's 'Elements of Machine Design.'

114. **Stress-strain Diagram.**—The relation between the elongation and the load producing it can be conveniently exhibited in the form of a diagram. Let the stress be represented by an ordinate $O y$ drawn vertically (not shown on the diagram), and

TABLE V.—ULTIMATE AND ELASTIC STRENGTHS OF MATERIALS, AND COEFFICIENTS OF ELASTICITY IN LBS. PER SQUARE INCH.

Material	Breaking Strength		Elastic Strength		Coefficient of Elasticity	
	Tension	Pressure	Tension	Pressure	Direct, E	Transverse, C
Cast iron	30,500 17,500 10,800	130,000 95,000 50,000	10,500	21,000	23,000,000 17,000,000 14,000,000	7,600,000 6,300,000 5,000,000
Wrought-iron bars	67,000 57,600 33,500	50,000	30,000	30,000	31,000,000 29,000,000 27,000,000	10,500,000
Iron boiler-plates	47,000	—	24,000	24,000	26,000,000	14,000,000
Steel plates, ¼ per cent. carbon	65,000	—	42,000	38,000	31,000,000	
,, ½ ,, ,,	78,000	—	47,000	49,000	31,000,000	13,000,000
,, 1 ,, ,,	110,000	—	67,000	71,000	31,000,000	
Cast steel, untempered	150,000 120,000 84,000	—	80,000	80,000	30,000,000	12,000,000
,, tempered	—	—	190,000	190,000	36,000,000	14,000,000
Copper rolled plates	31,000	—	5,600	4,000	15,000,000	5,600,000
,, annealed wire	45,000	—	—	—	16,000,000	
,, hard drawn wire	58,000	—	—	—	17,000,000	
Brass From To	17,500 29,000	—	—	—	13,500,000	
Gun metal or bronze	52,000 27,000 23,000	—	6,200	—	13,500,000	
Delta metal, cast	36,000	—	17,000	—	12,000,000	
,, rolled	74,000	—	51,000	—	13,000,000	
Phosphor bronze	58,000	—	19,700	—	14,000,000	5,250,000

the corresponding extension be a line $y\,p$ drawn horizontally from y. The locus of the point p will be the stress-strain curve of the material. Stress-strain curves for a number of different materials subjected to tension are shown in figure 111.

It has been proposed to represent the comparative values of materials for constructive purposes by figures derived from their stress-strain curves. The work done in breaking a test piece, reckoned per cubic inch of volume, may be used. This is proportional to the area included between the base and the stress-strain curve. Tetmajer's 'value-figure' for a material is the product of the maximum stress and the elongation per unit length. It is the area of the rectangle formed by drawing from the final point of the stress-strain curve lines parallel to the axes. Of the materials represented in figure 111, 'Delta' metal and aluminium bronze have the highest 'value-figures.'

115. **Mild Steel.**—Figure 111 shows the stress-strain curve for mild steel, such as the material from which weldless steel tubes are made. The straight portion $O\,a$ represents the action within the elastic limit. If the load be increased beyond that represented by a, the extension takes place at a more rapid rate, as shown by the slightly curved portion $a\,b$. At a point, b, somewhat above the elastic limit, a, a sudden lengthening of the bar takes place without any increase of load, this being represented by the portion $b\,c$ of the curve. The stress at which this occurs is called the *yield-point* of the material. On further increasing the load, extension again takes place, at first comparatively slowly, but afterwards more rapidly, until the maximum stress at the point d is reached. Under this stress the bar elongates until it breaks. If, however, the stress be partially removed after the maximum stress, d, is reached, as can be done in a testing machine, the curve falls gradually, as at $d\,e$, then more rapidly until fracture occurs at f. The elongation represented by the curve up to E takes place uniformly over the whole length of the bar, that represented by $e\,f$ only on a small portion in the neighbourhood of the fracture.

In wrought iron, the yield-point is not so distinctly marked as in mild steel; the stresses at the elastic limit and at breaking are less, the elongation before fracture is also less. The specific gravity of wrought iron and mild steel is, on an average, 7·7.

116. **Tool Steel.**—For a tool steel of good quality, containing about one per cent. carbon, the maximum stress may be much higher; the stress-strain curve takes the form shown in figure 111,

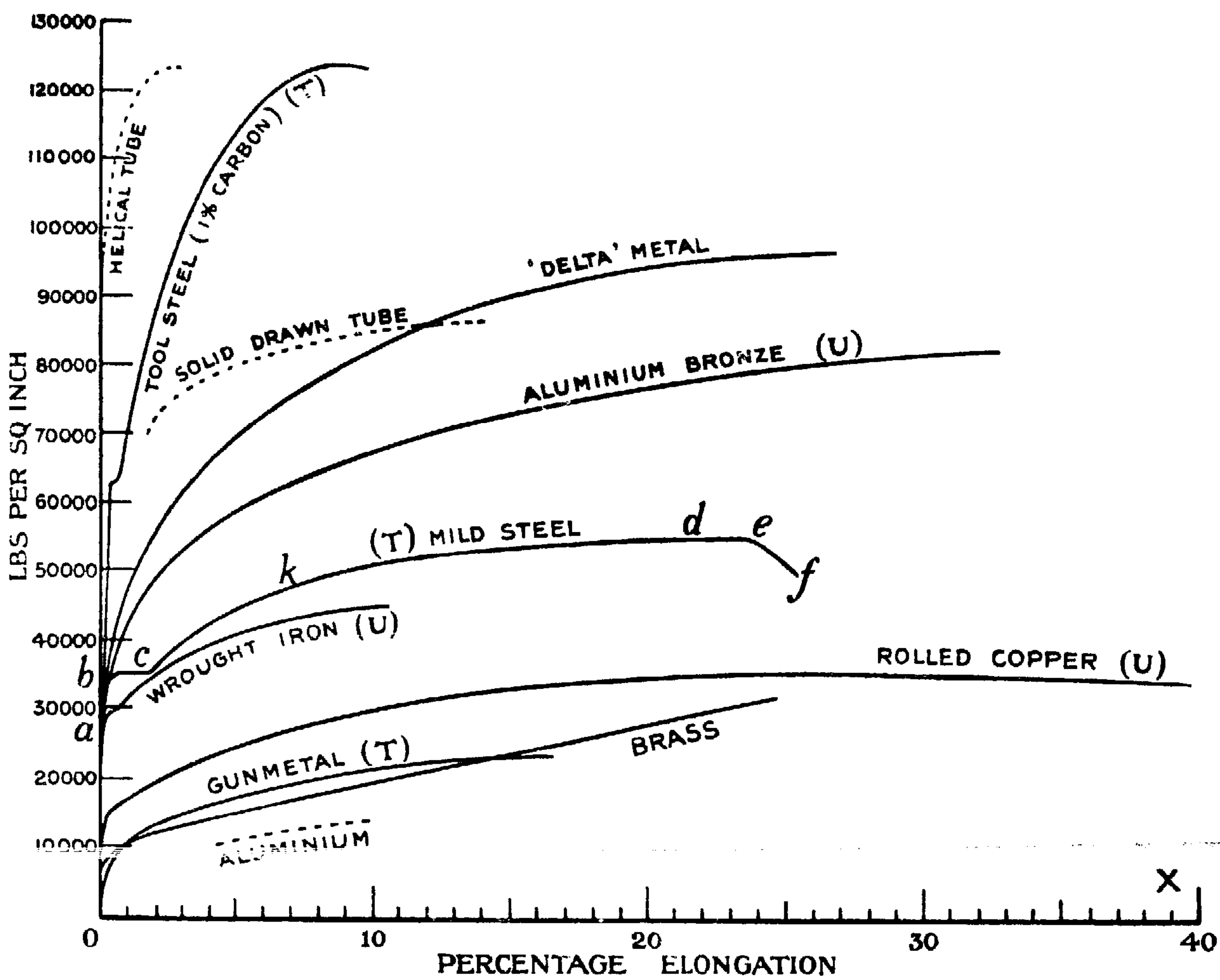


FIG. 111.

the extension being smaller, though the tenacity is very much greater, than that of mild steel.

117. **Cast Iron** has no well-defined elastic limit; in fact, the stress-strain curve is not straight for any part of its length, so that for cast iron the term 'elastic limit,' though often used, has no definite meaning.

118. **Pure Copper** varies greatly in tensile strength, according to the mechanical treatment to which it has been subjected. Rolling and wire-drawing both increase its tenacity. The stress-strain curve for rolled copper (fig. 111) is from Professor Unwin's 'The Testing of Materials of Construction.'

119. **The Alloys of Copper** with other metals form a most

important series. Their mechanical properties are most fully discussed in Professor Thurston's 'Brasses, Bronzes, and other Alloys.'

Brass contains 66–70 per cent. copper, and 34–30 per cent. zinc; sometimes a little lead. The stress-strain diagram (fig. 111) shows that the stress at the elastic limit is very low in comparison with the ultimate breaking stress.

Gun-metal is an alloy of copper and tin. The stress-strain diagram (fig. 111) is from a metal containing 98 per cent. copper, 2 per cent. tin.

Ternary alloys of copper, zinc, and tin have been exhaustively investigated by Professor Thurston. He finds the best proportion, when toughness as well as tenacity is important, is copper 55, tin 0·5, zinc 44·5.

Aluminium Bronze.—Copper and aluminium form a most useful series of alloys. The stress-strain curve (fig. 111) is from an alloy containing about 10 per cent. aluminium; it shows clearly the great strength and ductility of the material.

Alloys containing a much larger proportion of aluminium are valuable where lightness is the first consideration, but since they possess little strength and ductility, they can only be sparingly used in structural work.

Delta metal is a copper-zinc-iron alloy, which can be cast and worked hot or cold. It possesses great strength and ductility, as is shown by the stress-diagram (fig. 111) from a bar ·79 sq. in. sectional area, tested by Mr. A. S. E. Ackermann at the Central Technical College.

120. **Aluminium.**—A specimen of squirted aluminium, containing 98 per cent. of the pure metal, was tested at the Central Technical College by Mr. Ackermann; the tenacity was 6·32 tons per sq. in.; the elongation in 8″ was 1·12″, of which ·53″ was in the immediate neighbourhood of the fracture; the general elongation may, therefore, be taken as 10 per cent. For comparison this result is plotted in figure 111.

Pure aluminium has not sufficient strength and toughness to be of much value as a structural material, though its lightness as compared with other metals is a desirable quality. Some alloys, containing a small percentage of aluminium, possess great strength,

but they are, of course, heavy. It remains to be seen whether a strong alloy, containing a large percentage of aluminium, and therefore light, can be discovered. Such an alloy may possibly be of value in cycle making.

The specific gravity of sheet aluminium is 2·67, of mild steel 7·7.

121. **Wood** is not so homogeneous as most metals ; it is, as a rule, much stronger along than across the grain. The fact that wood joints are generally of low efficiency is against its use in tension members of a frame. For compression members, where there is [illegible] of strength at the joints, it may be used with advantage in some cases, its compressive strength (see Table VI.) being not much inferior, weight for weight, to that of the metals. In beams of short span subjected to bending, it is, in some important cases, immensely superior, weight for weight, to metal. The strength of a rectangular beam is proportional to its width, the *square* of its depth, and the strength of the material from which it is made (sec. 94), *i.e.* proportional to $b z^2 f$. If beams of equal weight be made from wood and steel, the width b being the same in both, the depth d of the wood beam will be greater than that of the steel beam ; and the product $z^2 f$ will be much greater for the wood than the steel beam.

The rim of a bicycle wheel is subjected to compression and bending (sec. 255). Since its width must be made to suit the tyre, a wood rim will be much stronger than a solid steel rim of the same weight ; or, for equal strengths, the wood rim will be the lighter. A *hollow* steel rim will possibly be stronger than a wood rim of equal weight.

Table VI., taken from Laslett's 'Timber and Timber Trees, gives the weights and strengths of a few woods.

122. **Raising of the Elastic Limit.**—Let a bar be subjected to a stress—represented by the point k (fig. 111)—considerably above its elastic limit. If the load be removed and the bar be again tested, it will be found that it is elastic up to a stress as high as that indicated by k. Thus the elastic limit in tension of a material like mild steel can be raised by simply applying an initial stress a little above the limit required.

An important application of this principle occurs in the case

TABLE VI.

SPECIFIC GRAVITY AND STRENGTH OF WOODS.

Name of wood	Specific gravity, water being taken 1·000	Transverse load on pieces 2″ × 2″ × 72″	Tensile stress on pieces 2″ × 2″ × 30″	Vertical stress on pieces 2″ × 2″ × 2″
		lbs.	lbs. per sq. in.	lbs. per sq. in.
Ash, English	·736	862	3,780	6,963
,, American	·480	638	5,495	5,494
Elm, English	·558	393	5,460	5,785
,, Canadian	·748	920	9,182	7,418
Fir, Dantzic	·582	877	3,231	7,104
,, Spruce, Canada	·484	670	3,934	4,852
Kauri, New Zealand	·530	816	4,543	5,880
Larch, Russian	·646	626	4,203	5,985
Oak, English	·735	776	7,571	7,640
,, French	·976	878	8,102	7,942
,, White, American	·983	804	7,021	6,964
Pine, Yellow	·554	505	2,027	4,172
,, Pitch, American	·659	1,049	4,666	6,462

of Southard's twisted cranks. Here the cranks are given a considerable initial twist in the direction in which they are strained while driving ahead ; their strength is considerably increased thereby. A twist (sec. 109) is equivalent to a direct pull along certain fibres, and a direct compression along other fibres at right angles. The initial twist in Southard's crank is, therefore, equivalent to raising the elastic limit of tension of the fibres under tensile stress, and the elastic limit of compression of the fibres under compressive stress.

123. **Complete Stress-strain Diagram.**—The complete stress-strain diagram of a material should extend below the axis OX; in other words, it should give the contractions of the bar under compressive stresses, as well as elongations under tensile stresses. Figure 112 represents such a curve, the point a denoting the elastic limit in tension, and b the elastic limit in compression. If the bar has had its elastic limit in tension raised artificially to the point k (fig. 111), it is found experimentally that the elastic limit in compression has been lowered, and thus the new stress-strain curve would be somewhat as represented in figure 113.

These considerations, when applied to the case of Southard's cranks, detract from the value of the initial twist. The line $t\,t$ (fig. 109), which is the tension line when the rider is pedalling ahead, has had its elastic limit in tension artificially raised, and its elastic limit in compression artificially lowered by the initial twist. When back-pedalling, $t\,t$ becomes the compression line. A twisted crank is therefore weaker for back-pedalling than an untwisted crank of the same material.

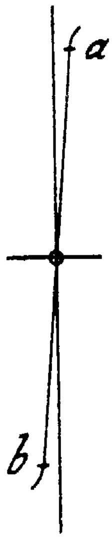


FIG. 112.

FIG. 113.

124. **Work done in Breaking a Bar.**—A material that gives very little extension before breaking is said to be wanting in *toughness*, and is not so suitable for structural purposes as a material with a larger extension. The total elongation of a material is usually expressed as a percentage of its original length. If the actual instead of the percentage elongations be set off horizontally (fig. 111), the area included between the stress-strain curve, its end ordinate, and the axis $O\,X$, represents the work done in breaking the bar. A bicycle is a structure subjected not to steadily applied forces but to impact. The relative value of a hard and a tough material for resisting such straining actions may be illustrated by an example.

Example.—Take a material like hardened steel, elastic up to its breaking-point, so that its stress-strain diagram is as shown at figure 114. Let its breaking-stress be 60 tons per square inch, and $E = 12{,}000$ tons per square inch. Then the extension at breaking-point is

$$E = \frac{60}{12000} = \cdot 005.$$

If the original length of the bar be 10 inches, the total elongation $O\,x$ (fig. 114) will be $\cdot 05$ inches, and the work done will be the area of the triangle $O\,a\,x$,

$$= \tfrac{1}{2} \times 60 \times \cdot 05 = 1 \cdot 5 \text{ inch-tons.}$$

Take now a material like mild steel, and consider that its stress-strain curve is quite straight up to the yield-point b (fig. 115).

Let the yield-point occur at 15 tons per square inch ; then, taking E, as before, 12,000 tons per square inch, and the original length of the bar 10 inches, $O\,x$ will be ·0125 inches. The work done in stretching the bar up to the yield-point will be

$$\tfrac{1}{2} \times 60 \times \cdot 0125 = 0{\cdot}375 \text{ inch-tons.}$$

Consider both bars to be acted on by a force of impact equivalent to a weight of 10 tons falling through a height of $\frac{1}{5}$ inch. The work stored up in this falling weight will be

$$10 \times \tfrac{1}{5} = 2 \text{ inch-tons.}$$

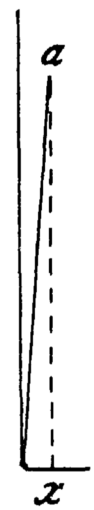


FIG. 114.

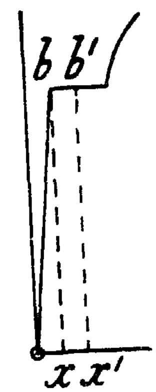


FIG. 115.

This must be taken up by the bar. But the work done in breaking the hard steel bar of high tenacity is only 1·5 inch-tons ; it would therefore be broken by such a live load. The mild-steel bar would be stretched an additional length, $x\,x_1$, until the total area, $O\,b\,b^1\,x_1$, was equal to 2 inch-tons. The area, $b\,b_1\,x_1\,x$, is therefore

$$2 - 0{\cdot}375 = 1{\cdot}625 \text{ inch-tons.}$$

The distance $x\,x_1$ will be

$$\frac{1{\cdot}625}{15} = \cdot 108 \text{ inch.}$$

Thus the only effect of the impulsive load on the mild steel bar is to stretch it a small distance, though the same load is sufficient to break the bar of much higher tensile strength but with little or no elongation before fracture.

The above examples show that the elongation before fracture of a material is almost as important as its breaking strength in determining its value as a material for bicycle building.

125. **Mechanical Treatment of Metals.**—The tenacity of a metal is almost invariably increased by rolling, or by drawing through dies. A metal to be drawn into wire or tube must be strong and ductile. The finest wire is made from a metal in which the ratio of the elastic to the ultimate strength is low. A metal with very high tenacity has not generally the ductility necessary for drawing into tubes or wire. The Premier Cycle Company,

instead of using drawn tubes, which must be made from a steel having a comparatively low tenacity, build up their tubes from flat sheets bent into spirals, each turn of the spiral overlapping the adjacent one, so that there are two thicknesses of plate at every part of the tube (fig. 116). A steel of much higher tenacity can

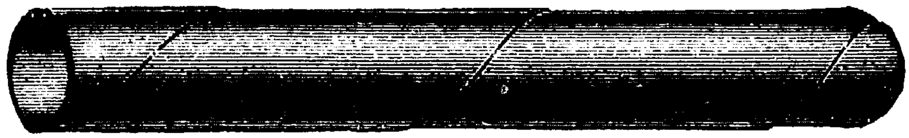


FIG. 116.

be used for this process than could be successfully drawn into tubes. These 'helical' tubes, therefore, have greater tenacity but less ductility than weldless steel tubes, as is shown by the comparative tests of helical and solid-drawn tubes 1 inch external diameter, recorded in Table VII. For comparison with other materials, the results of these tests are plotted on figure 111; the final points of the stress-strain diagrams being the only ones obtainable from the data, the curves are drawn dotted.

TABLE VII.

TENSILE STRENGTH OF HELICAL AND SOLID-DRAWN TUBES.

Description	Sectional area	Ultimate stress	Extension in 10 inches	Appearance of fracture
	Sq. in.	lbs. per sq. in.		
Helical 14A .	0·105	117,000	3·1	12 per cent. silky 88 per cent. granular
,, 20A .	0·107	122,000	1·5	Granular
,, 20C .	0·134	130,000	3·4	Granular
Solid-drawn C_1 .	0·106	80,000	18·7	Silky
,, H_1 .	0·106	94,000	8·0	Silky

126. **Repeated Stresses.**—If a bar be subjected to a steady load just below its breaking load, it will support it for an indefinite period provided the load remains constant, neither being increased or diminished. If the load is variable, however, the condition is quite different. Wöhler has shown that if the load vary from a maximum T_1 to a minimum T_2, fracture will occur when T_1 is less than the statical breaking load T, after a certain number of alterations from T_1 to T_2. The number of repetitions of the load

before fracture takes place depends not only on T_1 but on the difference $T_1 - T_2$, between the maximum and minimum loads. With a great range of stress the number of repetitions before fracture is less than with a smaller range.

A steel axle tested by Bauschinger, which had a statical tensile strength of 40 tons per square inch, stood at least two or three million changes of load before breaking, with the following ranges of stress :

Minimum stress tons per sq. in.	Maximum stress tons per sq. in.	Range of stress tons per sq. in.
−10·5	+10·5	21·0
0	19·7	19·7
20	32·1	12·1
40	40	0

A fuller discussion of this subject is given in Professor Unwin's 'Machine Design' and 'The Testing of Materials of Construction.'

The running parts of a bicycle—the wheels, chain, pedal-pins, cranks, and crank-axle—are subjected, during riding, to varying stresses. The range of stress on the spokes is probably small, so that a high maximum stress may be used without running any risk of fracture after the machine has been in use a considerable time. The stress on a link or rivet of the chain varies from zero, when on the slack side, to the maximum on the tight side. The double change of stress on the pedal-pins, cranks, and crank-axle takes place once during each revolution of the latter. A distance of 5,000 miles ridden on a bicycle geared to 60″ corresponds to 1,500,000 double changes of stress on the cranks and axle. If these be made light (see chapter XXX.), no surprise need be expressed if fracture occurs at any time, after having run satisfactorily for one or two years.

PART II

CYCLES IN GENERAL

PART II

CYCLES IN GENERAL

CHAPTER XIV

DEVELOPMENT OF CYCLES: THE BICYCLE.

127. **Introductory.**—Wheeled vehicles drawn by horses have probably been used by all civilised nations. The *chariot* of the ancients was two-wheeled, the wheels revolving upon the axle. Coming down to later times, the *coach*, a covered vehicle for passengers, appears to have been first made in the thirteenth century, the earliest record relating to the entry of Charles of Anjou and his queen into Naples in a small *carretta*. The first coaches in England are said to have been made by Walter Rippon for the Earl of Rutland in 1555, and for Queen Elizabeth in 1564. The weight of these early coaches was probably so great that for centuries it seemed utterly impracticable to make a vehicle that could be propelled by the rider. With the growth of the mechanical arts, at the beginning of this century, more attention was given to the subject. Starting from the four-wheeled vehicles drawn by a horse, the most obvious step towards getting a pedo-motive vehicle was to make one of the axles cranked, and let the rider drive it either direct or by a system of levers, the wheels being rigidly fastened to the ends of the axle. Such a cycle is illustrated in figures 117, 118. If this cycle had to travel in straight lines or curves of large radius, as on a railway, it might have been, apart from its weight, fairly satisfactory. A grave mechanical defect was that in moving round a sharp curve one or both driving-wheels slipped, as well as rolled, on the ground, with a corresponding waste of energy in friction.

The first attempts at overcoming this difficulty consisted in fastening only one wheel rigidly to the driving-axle, the other running freely. This gave, however, a machine which did not always respond to the steering gear as the rider wished ; in fact, while a driving effort was being exerted, the machine tended to turn to the side opposite to the driving-wheel (see chap. xviii.). The introduction of the differential driving-axle, which allows both

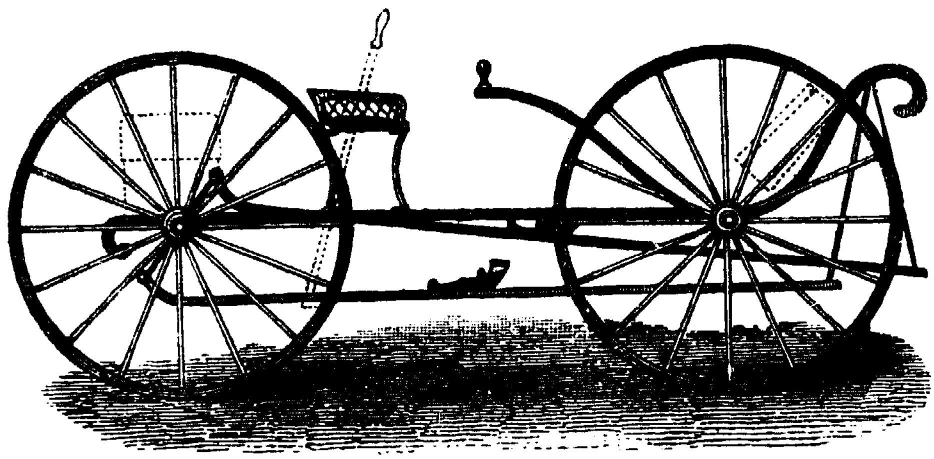


FIG. 117.

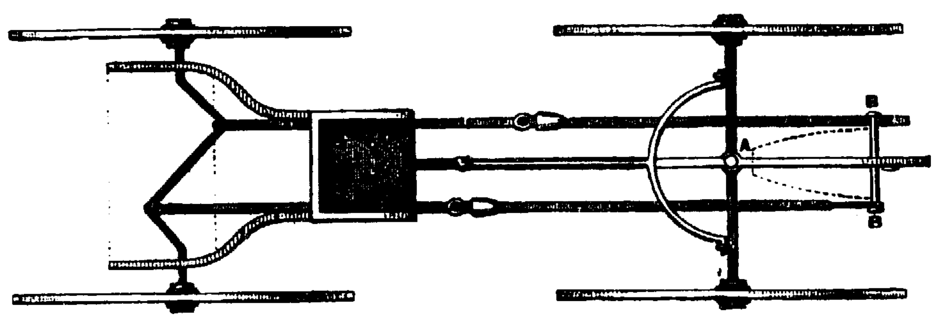


FIG. 118.

wheels to be driven at different speeds, overcame this difficulty completely without introducing any new ones.

The weight of the four-wheeler, and even of the three-wheeler, was, however, so great that it was not in this direction that cycles were at first developed. A wooden frame for supporting two wheels was, of course, much lighter than one for three wheels ; for this reason principally, bicycles were brought to a fair degree of perfection before tricycles. The use of steel tubes for the various parts of the frame made it possible to combine the strength and lightness necessary for a practicable cycle, and laid on a sure basis the foundations of the cycle-making industry.

Without attempting to give an exhaustive history of the de-

velopment of bicycles and tricycles, a short account of the various types that have from time to time obtained public favour may be given here.

128. **The Dandy-horse.**—Figure 119 may be taken as the first velocipede man-motor carriage. This was patented in France in 1818 by Baron von Drais. In 'Ackermann's Magazine,' 1819, an account of this pedestrian hobby-horse is given. "The principle

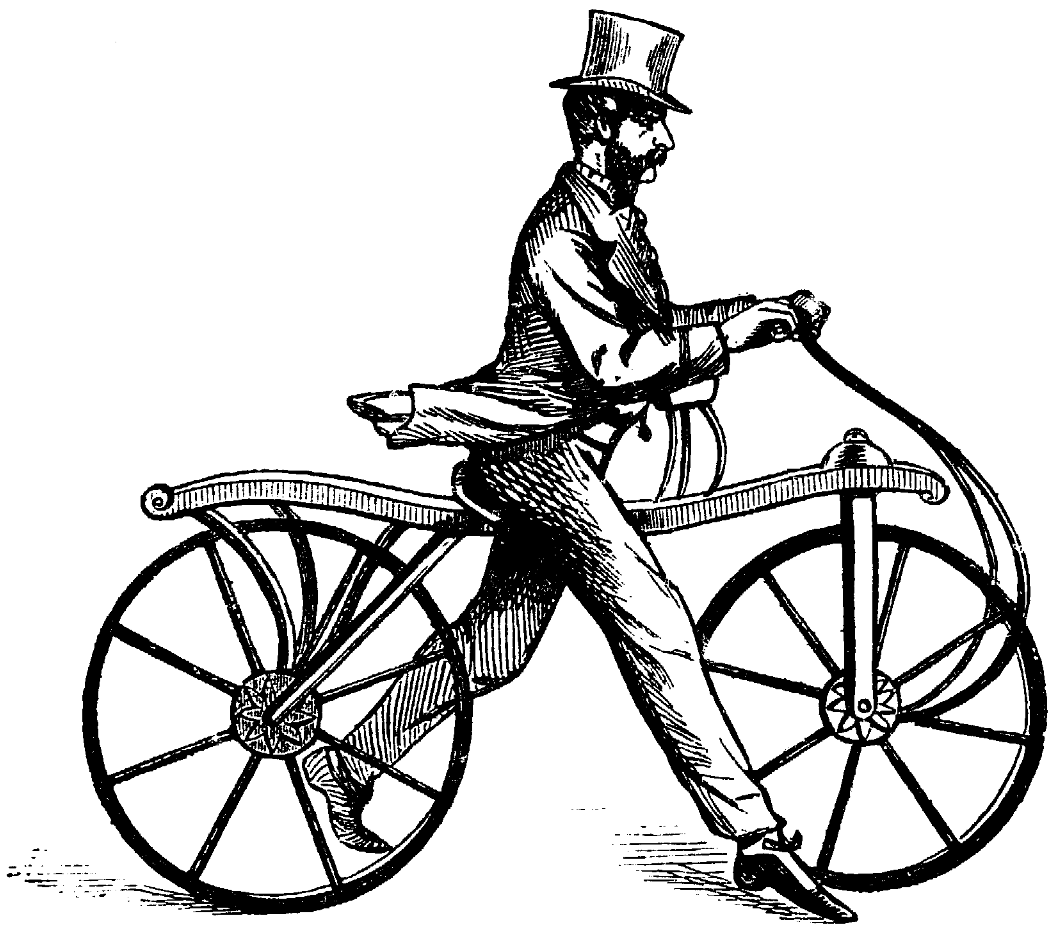


FIG. 119.

of the invention consists in the simple idea of a seat upon two wheels propelled by the two feet acting on the ground. The riding seat or saddle is fixed upon a perch on two short wheels running after each other. To preserve the balance a small board covered and stuffed is placed before, on which the arms are laid, and in front of which is a little guiding pole, which is held in the hand to direct the route. The swiftness with which a person well practised can travel is almost beyond belief, 8, 9, and even 10 miles may, it is asserted, be passed over within the hour on good level ground."

129. **Early Bicycles.**—Messrs. Macredy and Stoney, in 'The Art and Pastime of Cycling,' write: "To Scotland, it appears, belongs the honour of having first affixed cranks to the bicycle;

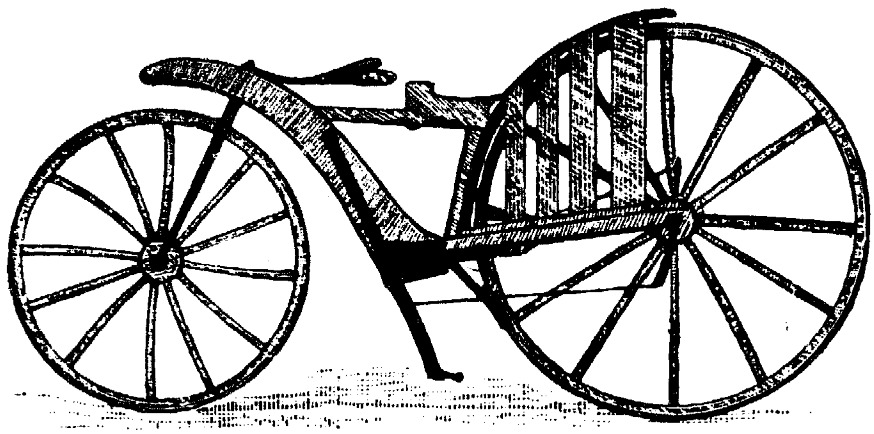


FIG. 120.

and, still stranger to relate, it was not to the 'hobby-horse,' but to a low-wheeled rear-driver machine, the exact prototype of the present-day Safety. The honour of being the inventor has now been fixed on Kirkpatrick M'Millan, of Courthill, Dumfries-

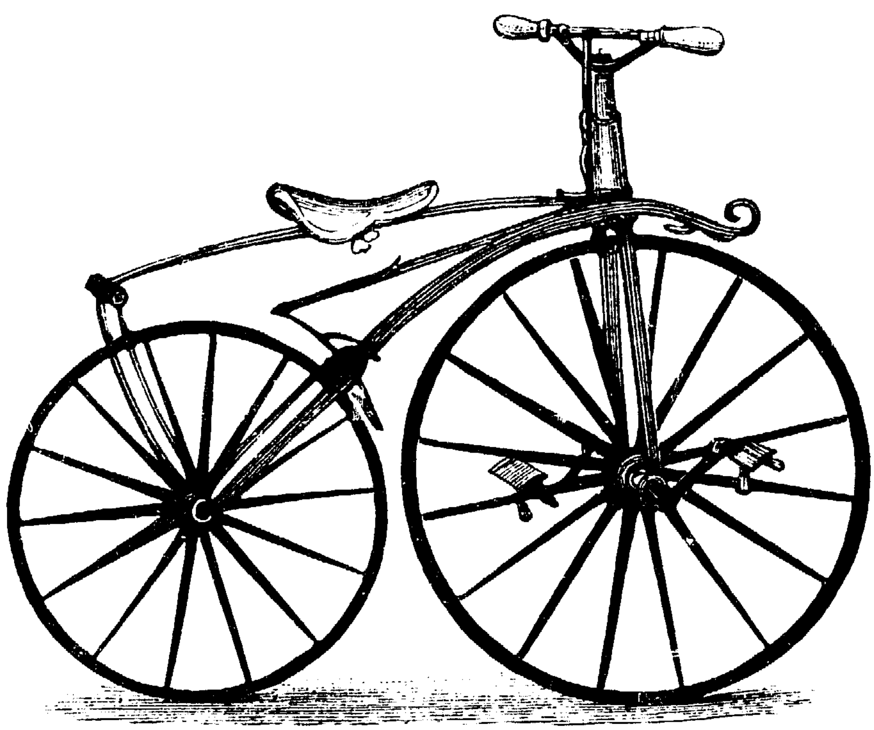


FIG. 121.

shire, though prior to 1892 Gavin Dalzell of Lesmahagow was the reputed inventor. It seems, however, that Dalzell only copied and probably improved on a machine which he saw with M'Millan.

M'Millan first adapted crank-driving to the 'hobby-horse' about the year 1840, and it was not earlier than 1846 that Dalzell built a replica of M'Millan's machine, a woodcut of which we reproduce (fig. 120). M'Millan is said to have frequently ridden from Courthill to Dumfries, some fourteen miles, to market on his machine, keeping pace with farmers in gigs." Figure 121 illustrates the 'French' bicycle or Bone-shaker,' which was in popular favour during the sixties. The improvement on the Dandy-horse consisted principally in the addition of cranks to the front wheel, so that the rider was supported entirely by the machine.

In 'Velocipedes, Bicycles, and Tricycles,' published by George Routledge & Sons in 1869, descriptions and illustrations of the bicycles, tricycles, and four-wheelers then in use are given. The concluding paragraph of this little book may be quoted: "Ere I say farewell, let me caution velocipedists against expecting too much from any description of velocipede. They do not give power, they only utilise it; there must be an expenditure of power to produce speed. One is inclined to agree with the temperate remarks of Mr. Lander, C.E., of Liverpool, rather than with the extravagant enthusiasm of American or French riders. As a means of healthful exercise it is worthy of attention. Certainly not more than forty miles in a day of eight hours can be done with ease; Mr. Lander thinks only thirty. If this is correct, it does not beat walking, though velocipedists affirm that double the distance can be done with ease. Much will and must depend on the skill of the rider, the state of the roads, and the country to be travelled."

130. **The Ordinary.**—What has since been called the 'Ordinary' bicycle came into use early in the seventies. Figure 122 illustrates one made by Messrs. Humber & Co., Limited. The great advance on the bicycle illustrated in figure 121 consisted mainly in the use of indiarubber tyres, thus diminishing vibration and jar, and consequently diminishing the power necessary to propel the machine. As a direct consequence of this, a larger driving-wheel could be driven with the same ease as the comparatively small driving-wheel of the French bicycle. The design of the 'Ordinary' is simplicity itself, and it still remains the embodiment of grace and elegance in cycle construction, though superseded by its more speedy rival, the rear-driving Safety. The

motive power of the rider is applied direct to the driving-wheel; wheel, cranks and pedal-pins forming one rigid body. In this respect it has the advantage over bicycles of later design, with gearing of some kind or other between the pedals and driving-wheel.

In the 'Ordinary' the mass-centre of the rider was nearly directly over the centre of the wheel, so that any sudden obstruction to the motion of the machine frequently had the effect of sending

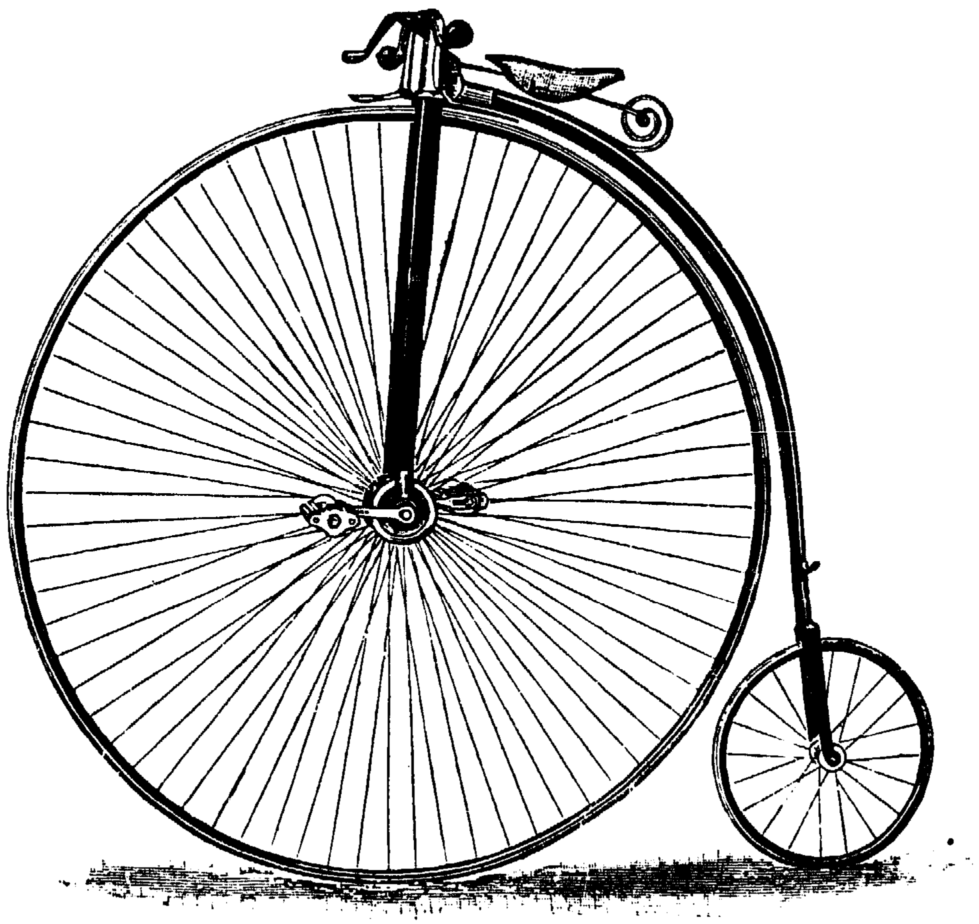


FIG. 122.

the rider over the handle-bar. This element of insecurity soon led to the introduction of other patterns of bicycles.

131. **The 'Xtraordinary'** (fig. 123), made by Messrs. Singer & Co., was one of the first Safety bicycles. The crank-pin was jointed to a lever, one end of which vibrated in a circular arc (being suspended by a short link from near the top of the fork), the other end was extended downwards and backwards, and supported the pedal. A smaller wheel could thus be used, and the saddle placed further back than was possible in the 'Ordinary.'

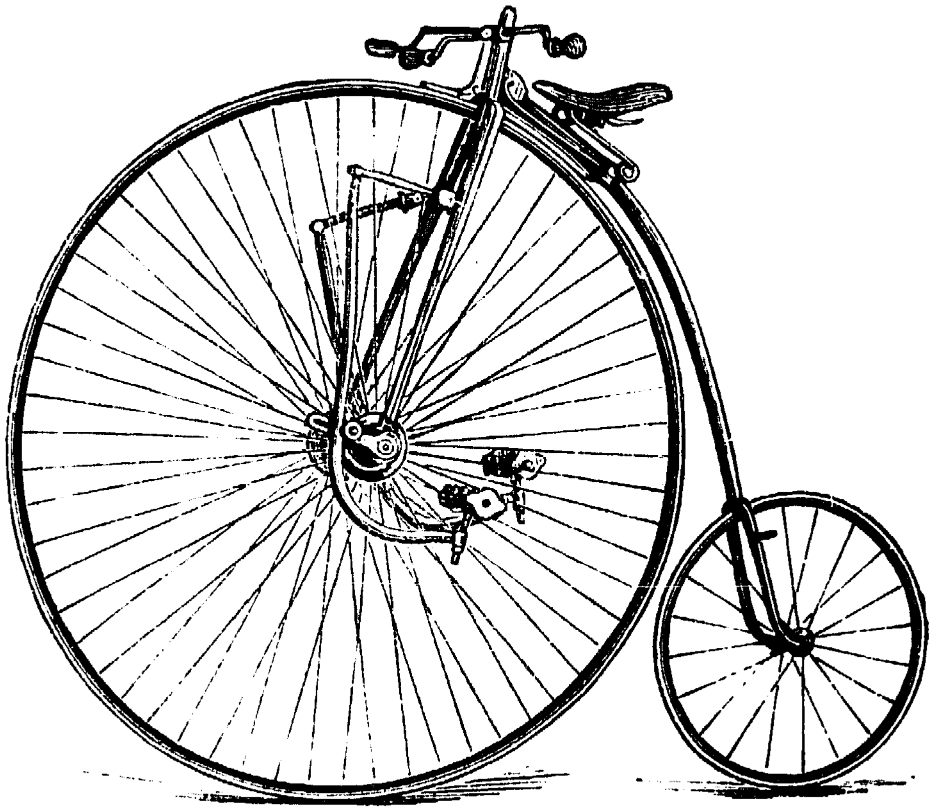


FIG. 123.

132. **The Facile.**—In the 'Facile' bicycle a smaller driving-wheel was used, and the mass-centre of the rider brought further behind the centre of the driving-wheel. This was accomplished

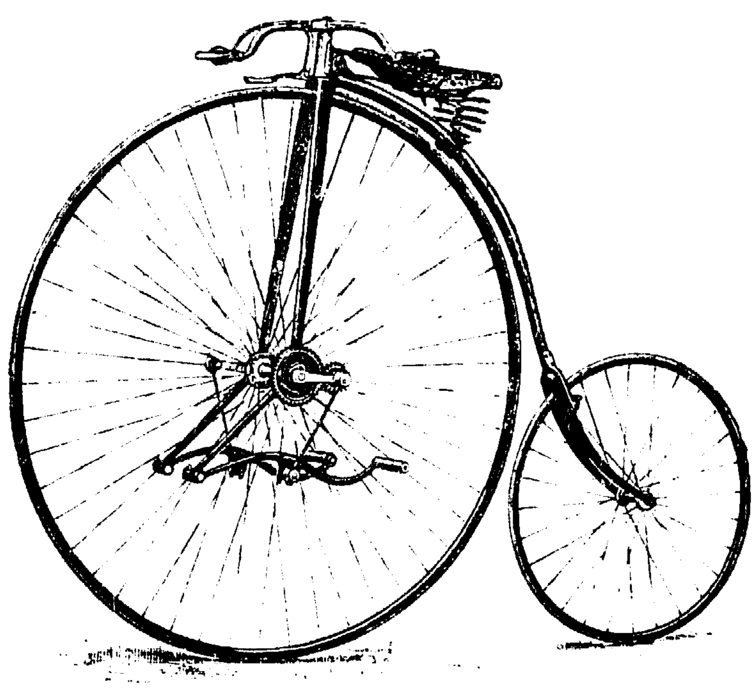


FIG. 124.

by driving the crank by means of a short coupling-rod from a point about the middle of a vibrating lever; the end of

this vibrating lever forming the pedal. The fork of the front wheel was continued downwards and forwards to provide a fulcrum for the lever. The motion of the pedal relative to the machine was thus one of up-and-down oscillation in a circular arc, and was quite different from that of the uniform circular motion in the 'Ordinary.' From the position of the mass-centre of the rider relative to the centre of the driving-wheel, it is evident that this bicycle possessed a much greater margin of safety than the 'Ordinary.' Also, from the fact that the machine and rider offered a less surface to wind resistance, the machine was easier to propel under certain circumstances than the 'Ordinary.' In 1883, Mr. J. H. Adams rode 242½ miles on the road within twenty-four hours; this was at that time the best authentic performance on record.

133. **Kangaroo.**—Figure 125 illustrates the 'Kangaroo' type of front wheel crank-driven Safety introduced by Messrs. Hillman,

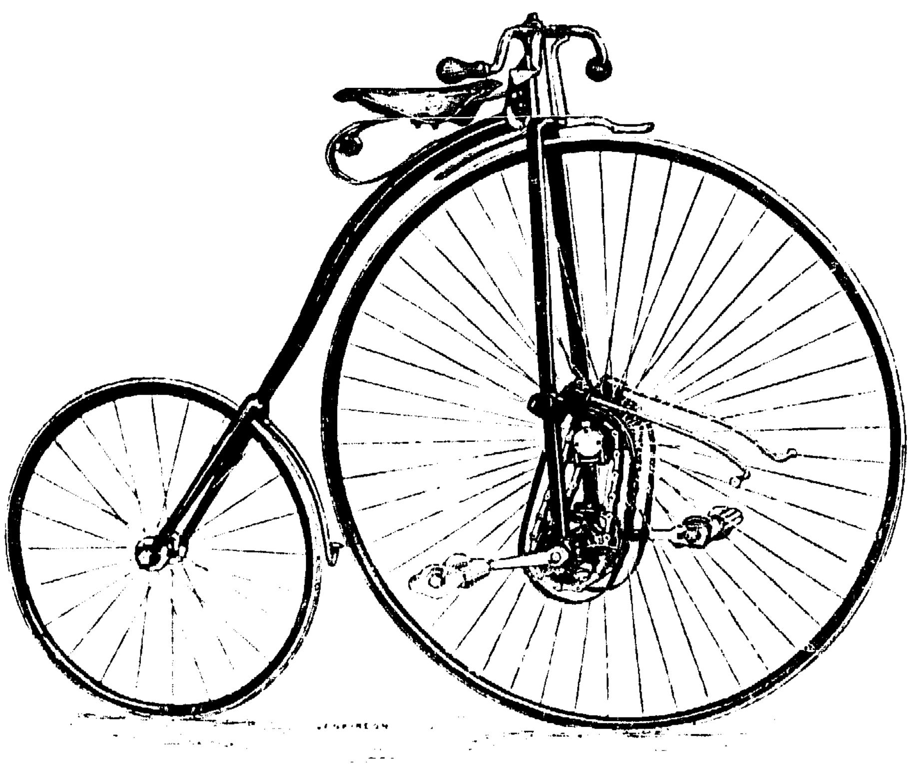


FIG. 125.

Herbert, and Cooper, 1884. A smaller driving-wheel is used than in the 'Ordinary'; the crank-axle is placed beneath and a little behind the centre of the driving-wheel. The crank-axle is divided into two parts, since its axis passes through the driving-wheel; the front-wheel fork is continued downwards to support the crank-

axle bearings; the motion of each portion of the crank-axle is transmitted by chain-gearing to the driving-wheel. In a 100-mile road race on September 27, 1884, organised by the makers of the machine, the distance was covered by Mr. G. Smith in 7 hours 7 minutes and 11 seconds, the fastest time on record for any cycle then on the road.

A geared dwarf bicycle is superior to an 'Ordinary' in two important respects, which more than compensate for the friction of the extra mechanism. Firstly, the rider being placed lower, the total surface exposed by the machine and rider is much less, the air resistance is therefore less, this advantage being greatest at high speeds. Secondly, since the speeds of the driving-wheel and crank-axle may be arranged in any desired ratio, the speed of pedalling and length of crank can be chosen to suit the convenience of the rider, irrespective of size of driving-wheel; while in an 'Ordinary' the length of crank is less, and the speed of pedalling greater, than the best possible values.

As regards safety, the 'Kangaroo' is a little better than the 'Ordinary,' but not so good as the 'Rover' or 'Humber' Safety. Two serious defects, which ultimately made it yield in popular favour to the rear-driving Safety, existed. A narrow tread must be kept between the pedals, and the consequent narrow width of bearing of the crank-axle gives a bad design mechanically. Again, the chains, however carefully adjusted initially, will, after a time, get a trifle slack. In pressing the pedal downwards the front side of the chain is tight, but when the pedal is ascending, since it cannot be lifted direct by the rider, it is pulled up by the chain, the rear side of which gets tightened. This reversal, taking place twice every revolution, throws a serious jar on the gear. This defect cannot, as in the 'Humber' with only one driving-chain, be overcome by skilful pedalling.

134. **The Rear-driving Safety** was invented by Mr. H. J. Lawson in 1879, but it was a few years later before it was in great demand. The 'Rover' Safety (fig. 126), made by Messrs. Starley and Sutton in 1885, was the first rear-driving bicycle that attained popular favour. The cranks and pedals are placed on a separate axle, the motion of which is transmitted by a single driving-chain to the driving-wheel. This type is absolutely safe as regards headers

over the handle-bar. Compared with the 'Kangaroo' gearing, the single driving-chain is a great improvement, as its driving side

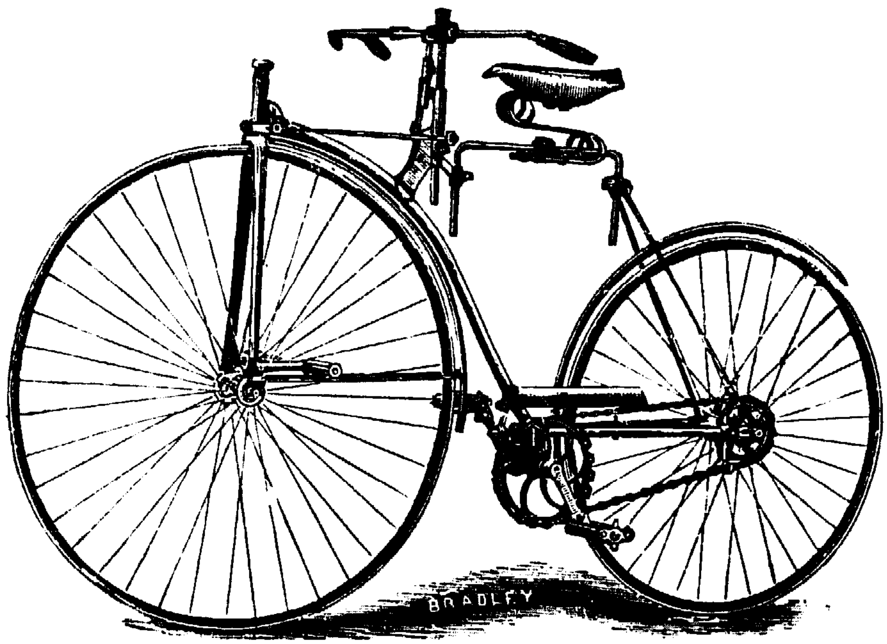


FIG. 126

may be kept tight continuously. The steering-head of the front wheel was vertical, and an intermediate handle-pillar was used, with coupling-rods to the front fork. In a later design (fig. 127)

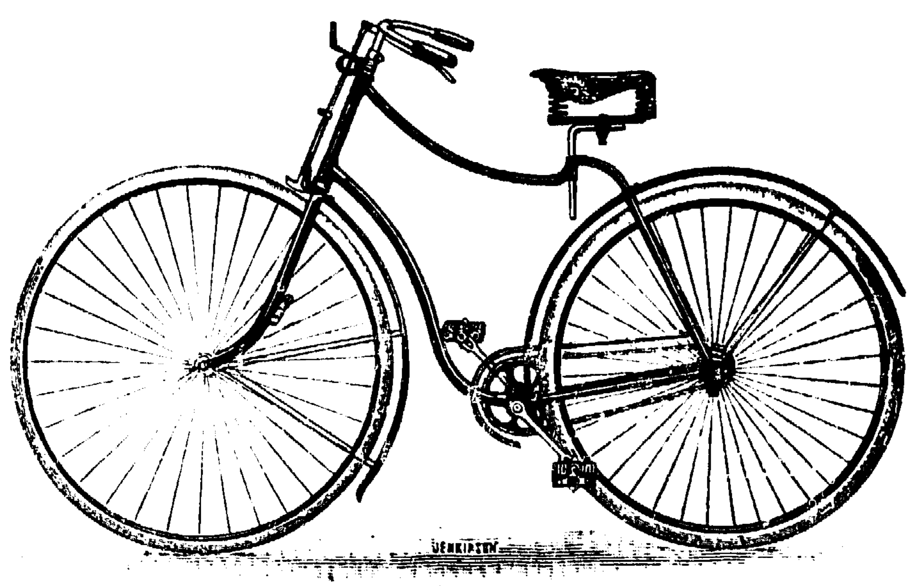


FIG. 127.

the front fork was sloped, and the steering made direct ; this machine thus formed the prototype of the modern rear-driving bicycle.

Figure 128 is an illustration of the 'Humber' Safety dwarf-roadster, made in 1885. In this all the arrangements of the 'Ordinary' may be said to be reversed; the proverbial Irishman's description of it being "The big wheel is the smallest, and the hind wheel is in front." The driving-wheel is changed from front to back, the small wheel is placed in front, and the mass-centre of the rider is brought nearer the centre of the rear wheel.

The 'Humber' Safety of 1885 is essentially the same machine as that in greatest demand at the present day. The improvements

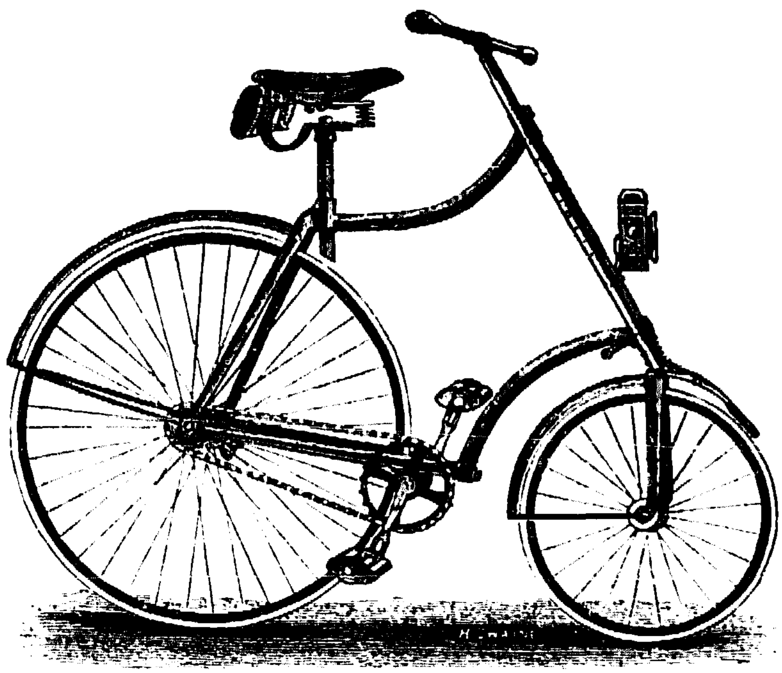


FIG. 128.

effected since 1885, though undoubtedly of very great practical importance, are merely improvements in details. Change in the relative size of the front and back wheels, different design of frame, and last, but not least, the introduction of pneumatic tyres, account for the different appearances of the earliest and latest Safeties.

Rear-driving Safeties were made by all the makers, the difference in bicycles by different makers being merely in detail. About this time (1886) the number of Safety bicycles made per annum began to increase very rapidly, while a few years later the number of 'Ordinaries' began to diminish.

135. **Geared Facile.**—The 'Facile' bicycle, with its small driving-wheel and direct link-driving from the pedal lever, necessitated

very fast pedal action on the part of the rider. The 'Geared Facile' (fig. 124) enabled the pedalling to be reduced to any desired speed. The connecting link in the 'Geared Facile' did not work directly on the driving-wheel, but the crank shaft ran loose co-axially with the driving-wheel, a sun-and-planet gear being inserted between the crank and the wheel. Figure 129

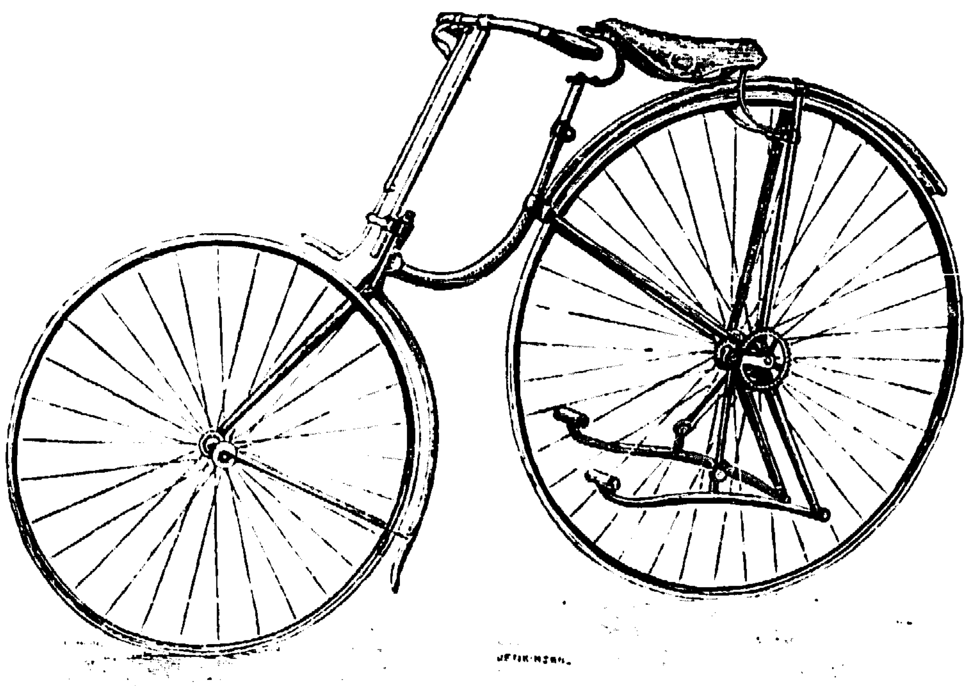


FIG. 129.

shows a 'Geared Facile' rear-driving bicycle, the usual sun-and-planet gear being modified to suit the altered conditions.

136. **Diamond-frame Rear-driving Safety.**—From the date of its introduction, the rear-driving Safety advanced steadily in popular favour until, in 1887, it was the bicycle in most general demand. In the preface to 'Bicycles and Tricycles of the Year 1888,' Mr. H. H. Griffin says: "We made careful inquiries of all those in a position to know as to the proportion of Dwarf Safeties and Ordinary bicycles, and were not a little surprised to hear that, taking the average through the trade, at least six Dwarf Safeties are made to one Ordinary." Up to the year 1890 the greatest possible variety existed in the frames of the rear-driving Safety, but they all agreed in having the distance between the rear and front wheels reduced to a minimum. The crank-bracket was placed just sufficiently in front of the driving-

wheel to have the necessary clearance, the steering-wheel sufficiently far in front to allow it in steering to swing clear of the pedals and the rider's foot. The down-tube, from the saddle to

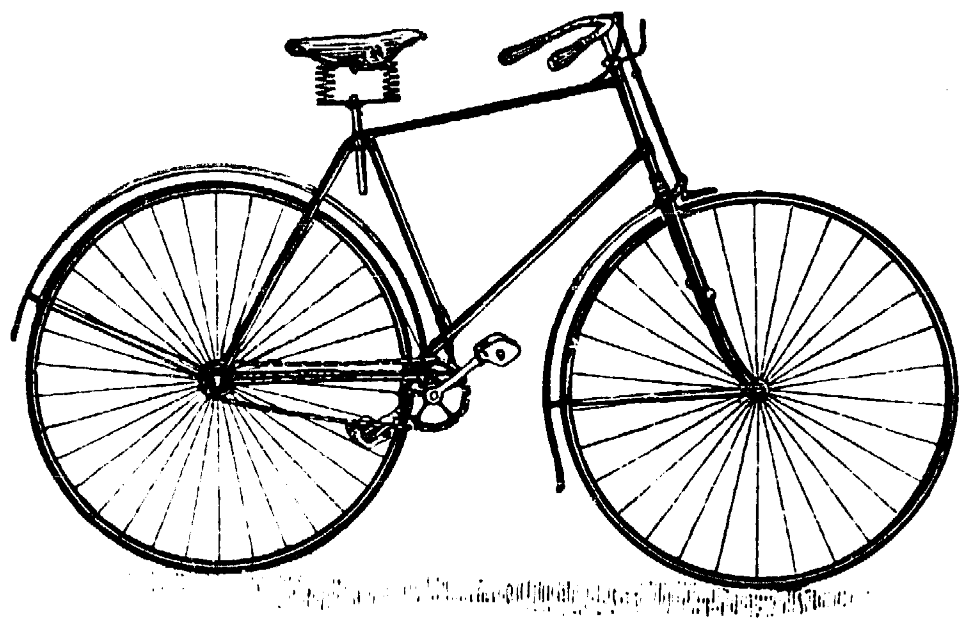


FIG. 130.

the crank-bracket, was usually curved, both in the diamond-frame and the cross-frame, or omitted altogether, as in the open-frame. Up till 1890 the nearest approach to the now universally adopted

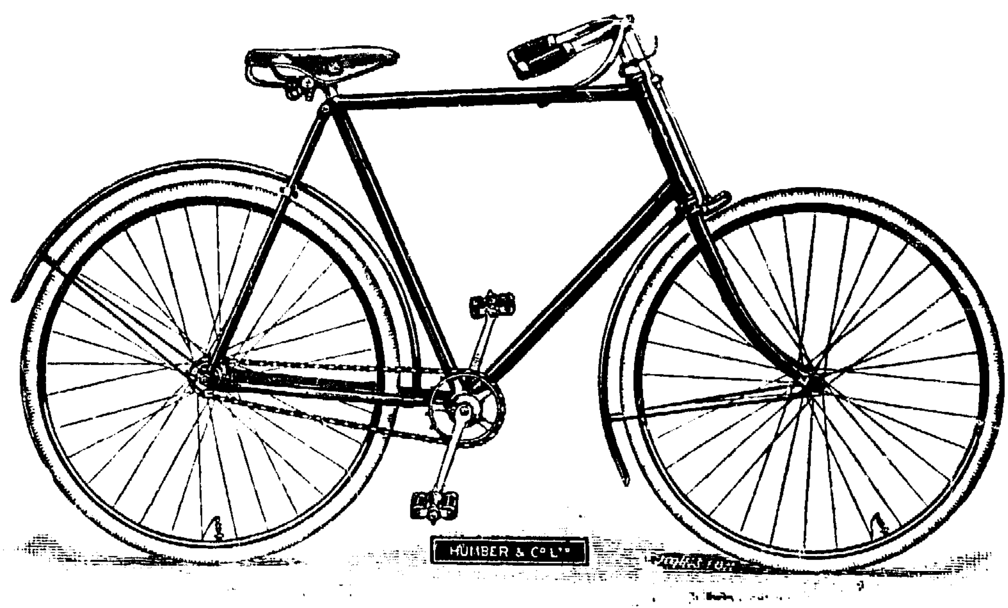


FIG. 131.

frame was that made by Humber & Co. (fig. 130). During these years the diamond-frame was being more and more generally adopted, and after Messrs. Humber introduced their rear-driving

Safety, with long wheel-base and diamond-frame (fig. 131), it became almost universal. By having several inches clearance between the crank-bracket and the driving-wheel, it was possible to use a straight tube from the saddle to the crank-bracket, while the long wheel-base rendered the steering more reliable. In the chapter on 'Frames' the reasons for the survival of the diamond-frame and the practical extinction of all others will be given.

137. **Rational Ordinary.**—The admirers of the 'Ordinary' bicycle were loth to let their favourite machine fall into disuse, and attempts were made to make it safer and more comfortable, by placing the saddle further behind the driving-wheel centre, by sloping the front fork, and by making the rear wheel larger than was usual in the 'Ordinary.' Such a machine was called a '*Rational* Ordinary.'

138. **Geared Ordinary** and **Front-driving Safety.**—In 1891, the Crypto Cycle Company—with whom Messrs. Ellis & Co., the makers of the 'Facile' and 'Geared Facile' had amalgamated—brought out a *Geared Ordinary*. This bicycle was in external appearance just like a 'Rational'; but the cranks, instead of being rigidly connected to the driving-wheel, drove the latter by means of an epicyclic gear (see sec. 306) concealed in the hub. The number of revolutions of the driving-wheel could thus be made greater than those of the crank; in fact, the machine could be geared up, just like a rear-driving Safety. The size of the driving-wheel being reduced, a front-driving Safety was obtained. Figure 132 shows the 'Bantam,' the latest development of the front-driver in this direction, with the front wheel 24 inches in diameter, and geared to 66 inches. The resemblance, in general arrangement at least, to the French bicycle (fig. 121) will be apparent, though as regards efficiency of action the two machines are as wide apart as the poles. Figure 243 shows

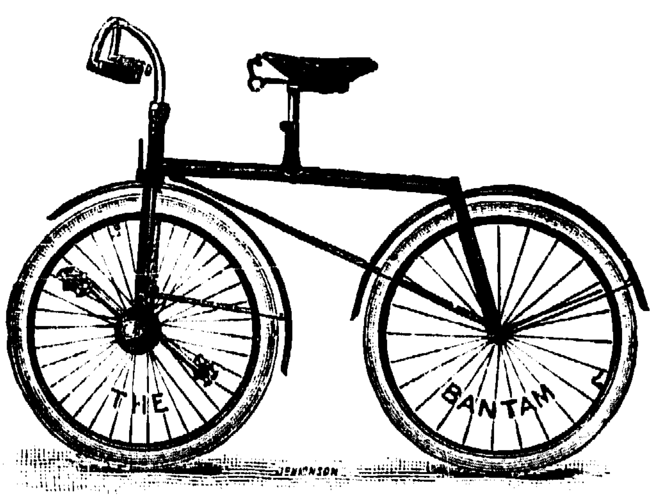


FIG. 132.

the 'Bantamette,' in which the frame is so arranged that the bicycle may be ridden by a lady.

139. **The Giraffe** and **Rover Cob.**—The 'Ordinary' had undoubtedly many good points which are missing in the modern Safety, among which may be mentioned greater lateral stability and steadiness in steering due to the high mass-centre. The 'Giraffe' (fig. 133), by the New Howe Machine Company, is a high-framed Safety, the saddle being raised as high as in the

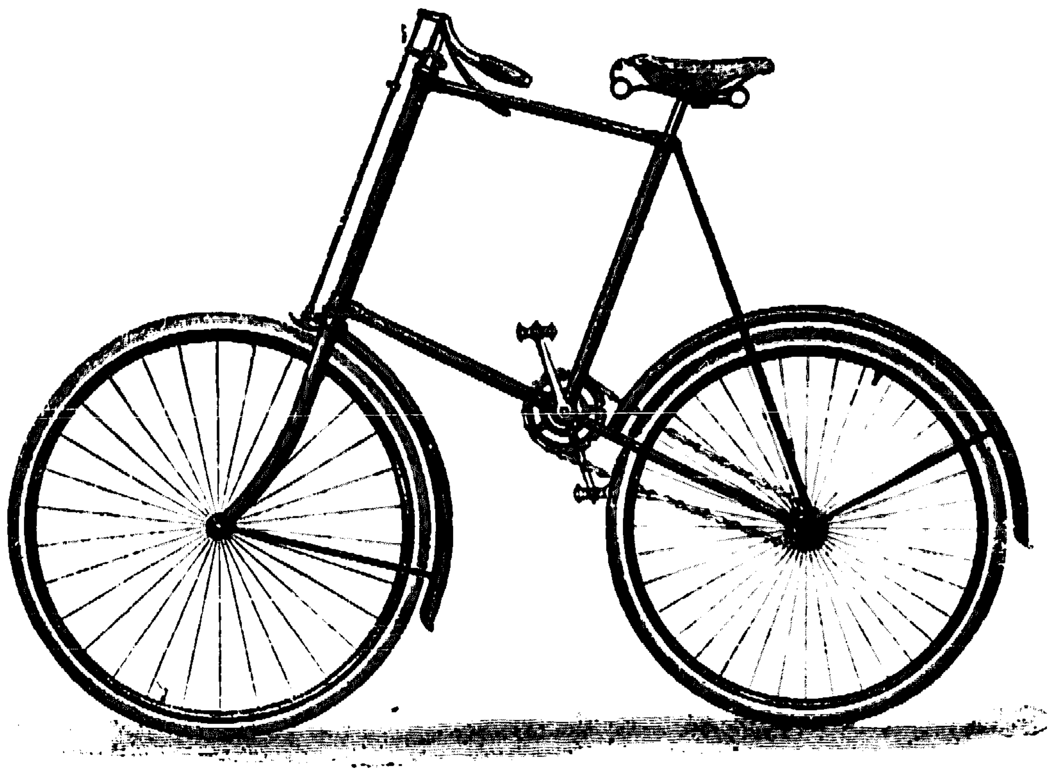


FIG. 133.

'Ordinary.' In the introduction to Leechman's 'Safety Cycling,' Mr. Henry Sturmey gives an enthusiastic account of the 'Giraffe, and a comparison with the low-framed Safety.

The 'Rover Cob' (fig. 134), made by Messrs. J. K. Starley & Co., is at the opposite extreme, the frame being made so low that the pedals will just clear the ground when rounding a corner at slow speed. It is intended for those who may have fear of falling; the mounting can be done by simply pushing off from the ground.

140. **Pneumatic Tyres.**—Whether judged by speed performances on the road or racing track, or from additional comfort and ease of propulsion to the tourist, the greatest advance in cycle construction due to a single invention must be credited to

Mr. James Dunlop, the inventor of the pneumatic tyre. A patent for a pneumatic tyre had been taken out by Thompson in 1848, but there is no record that he made a commercial success of his invention. In 1890, Mr. James Dunlop, of Dublin, made a pneumatic tyre for his son, and the results obtained by its use being so astounding, arrangements were very soon made for their manufacture. While in 1889 a pneumatic tyre was unheard of, at the Stanley Bicycle Club Show, November-December, 1891, from an analysis[1] of the machines exhibited, it appears that 40 per cent. of the tyres exhibited were *pneumatic*,

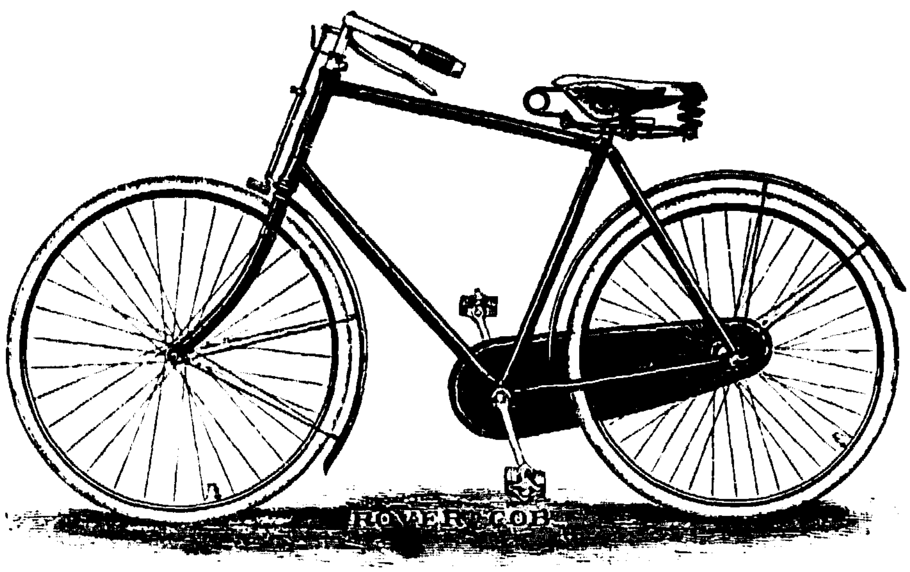


FIG. 134.

$32\frac{1}{2}$ per cent. *cushion*, $16\frac{1}{2}$ per cent. *solid*, 10 per cent. *inflated*, and the remainder, about 1 per cent., were classed as nondescript. In the above classification, under pneumatic tyres are included only those with a separate inner tube, the inflated being really single-tube pneumatic tyres. Cushion tyres were made and used as a kind of compromise between solids and pneumatics. The proportion of pneumatic tyres to the total has grown greater year by year, until now there is hardly a cycle made, for use in Britain at least, with any other than pneumatic tyres.

141. **Gear-cases.**—The most troublesome portion of a modern rear-driving bicycle is undoubtedly the chain and the accompanying gear. The chain, however well made originally, is found to stretch slightly under the heavy stresses to which it is subjected

[1] *The Cyclist's Annual and Year-book for* 1892.

in ordinary working. If the distance between the centres of the two chain-wheels—on the crank-axle and driving-wheel hub respectively—over which the chain passes is unalterable, the chain will ultimately get so slack that there will be a great risk of it overriding the teeth of the wheels, to the danger of the rider. All chain-driven cycles are consequently provided with some means of tightening the chain, *i.e.* of increasing the distance between the centres of the two chain-wheels. Again, in an exposed chain, it is practically impossible to lubricate perfectly the rubbing parts, very little of the oil applied to the outside surface finding

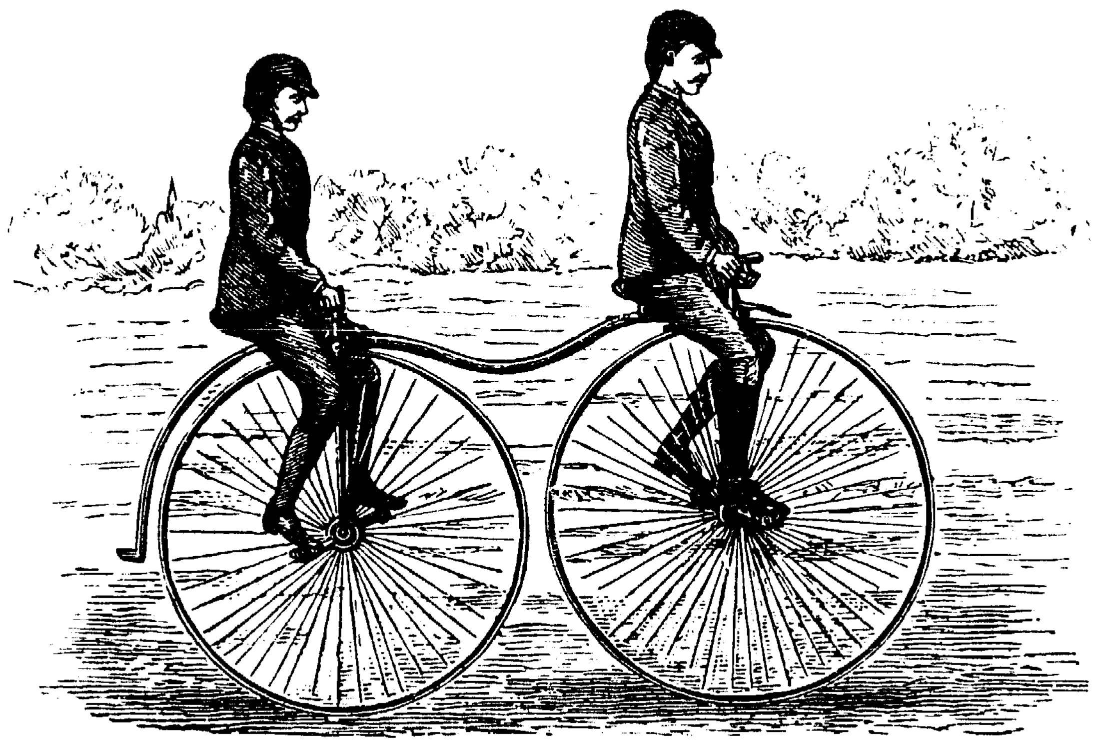


FIG. 135.

its way in between the rivet-pins and the blocks of the chain. Dust and grit from the road soon adhere to the chain and chain-wheel, so that the frictional resistance of the chain as it is wound on and off the chain-wheel is rapidly increased.

These considerations led Mr. Harrison Carter to introduce the *gear-case*, the function of which is to exclude dust and mud, and provide an oil-bath in which the lowest portion of the chain may dip. The reduction of frictional resistance is perhaps one of the least of the advantages pertaining to the use of the gear-case; one great advantage is that less trouble is given to the rider, and chain adjustments need not be made so frequently. In

fact, some makers claim that with an oil-tight gear-case the chain does not stretch perceptibly, and no chain adjustments are necessary. The author is not aware, however, that any maker has ventured to place on the market a bicycle with gear-case but no chain adjustment.

142. **Tandem Bicycles.**—When the success of the bicycle for a single rider was assured, attempts were soon made to make a bicycle for two riders. Figure 135 shows the 'Rucker' Tandem bicycle, made in 1884, one of the first successful tandem bicycles. This consists practically of two 'Ordinary' driving-wheels and forks connected together by a straight tubular backbone. At the front

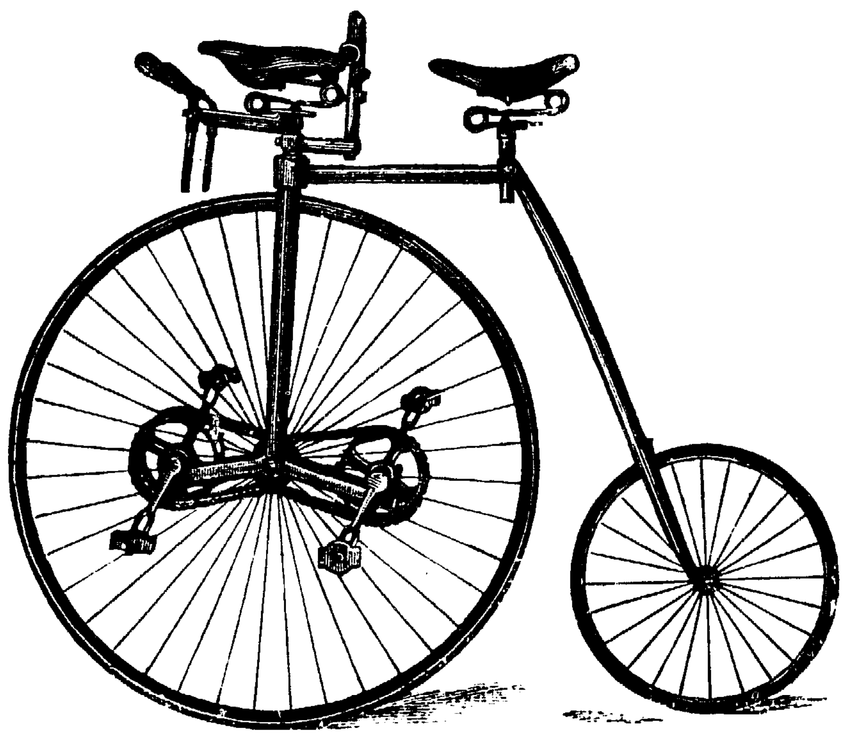


FIG. 136.

end of this backbone there is an 'Ordinary' steering centre ; at the other end it is connected to the head of the rear-wheel fork by a frame which permits it to twist sideways. Figure 136 shows a later andem bicycle, also made by Mr. Rucker—probably the first practicable machine of this type. It is practically a tandem 'Kangaroo.' In a paper on 'Construction of Cycles,' read before the Institution of Mechanical Engineers in 1885, Mr R. E Phillips says, "This tandem bicycle . . . eclipses the earlier, and bids fair to prove the fastest cycle yet produced. The weight is only 55 lbs., and it is, therefore, the lightest machine yet made to carry two riders"

Figure 137 shows a front-driving chain-driven Safety Tandem, made by Hillman, Herbert, and Cooper, 1887. Both riders drive the front wheel, and both wheels are moved in steering.

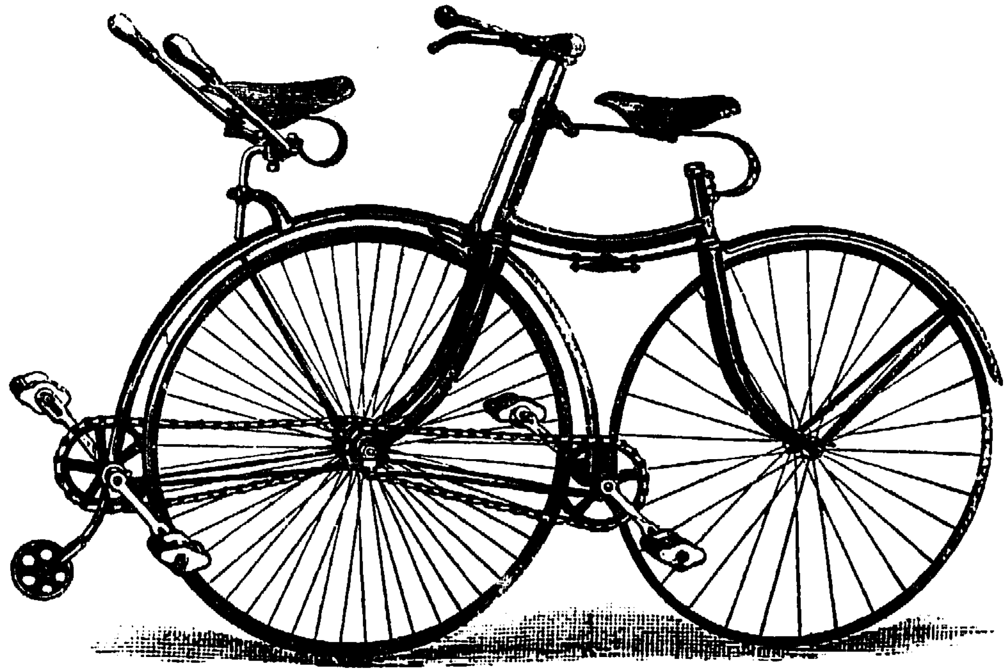


FIG. 137.

The 'Invincible' Tandem Safety (fig. 138), and the 'Ivel' Tandem Safety (fig. 139), which was made convertible so that it

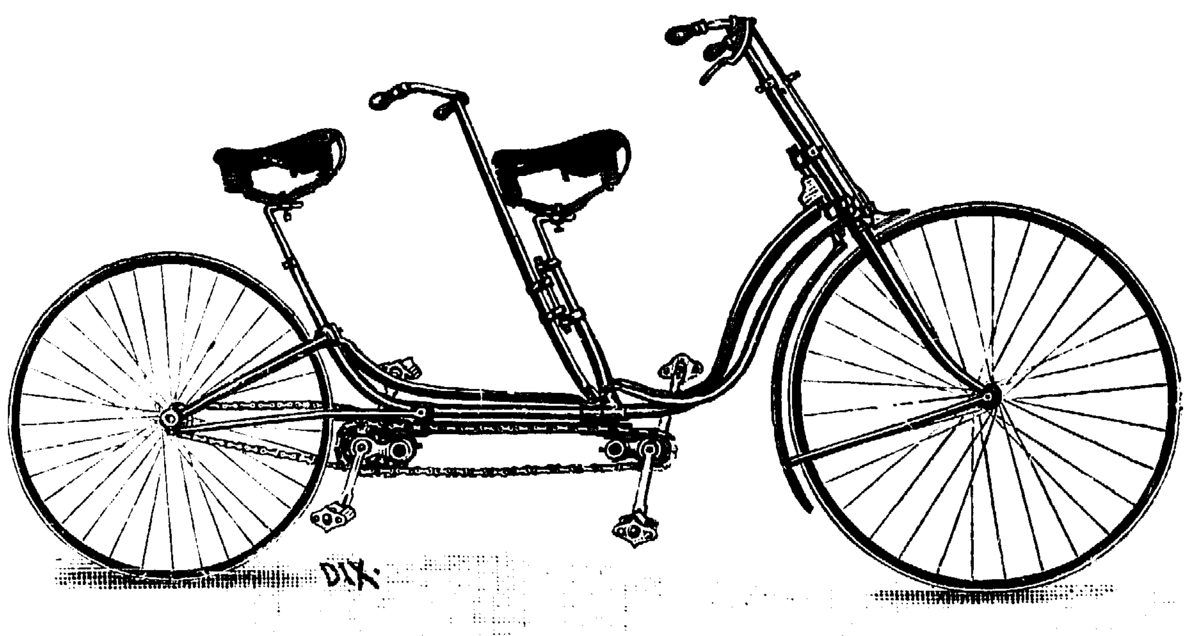


FIG. 138.

could be used as a single Safety, were among the first approximations to the present popular type of Tandem Safety, both riders

being placed between the wheels, and both driving the rear wheel. It will be noticed that the front crank-axle is connected by chain gearing to the rear crank-axle, the two axles rotating at the same speed ; the second chain passes over the larger wheel on the rear crank-axle and the chain-wheel of the driving-axle.

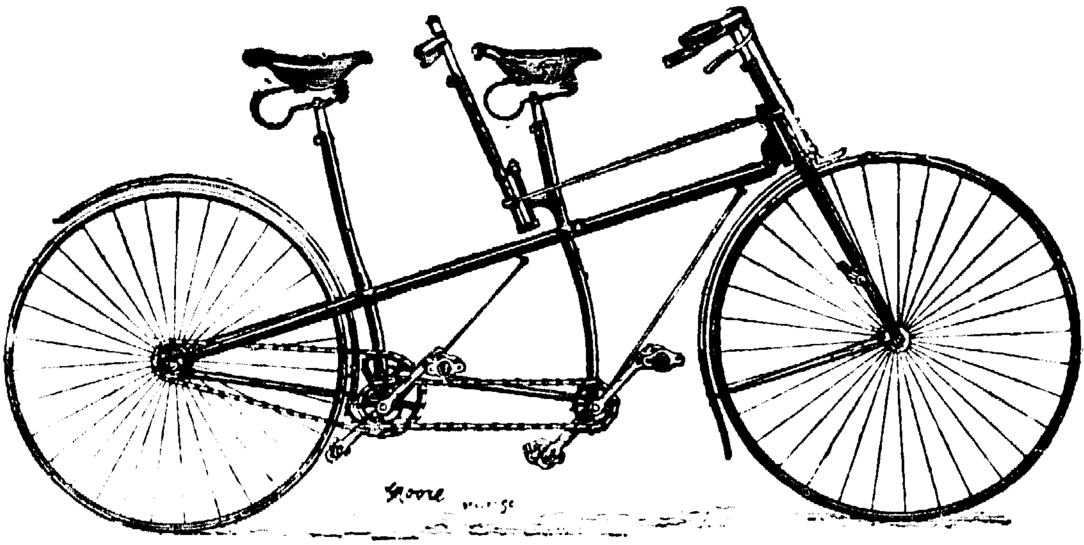


FIG. 139

Both riders have control of the steering, a light rod connecting the front fork to the rear steering-pillar. The long wheel-base of these bicycles adds to the steadiness of the steering at high speeds, since (see fig. 202), for the same deviation of the handle-bars, a machine with long wheel-base will move in a curve of larger radius than one with a shorter wheel-base. The distance between the wheel centres being much greater than in the single machine, the frame is subjected to very much greater straining actions, and imperfect design will be much more serious than in the single machine.

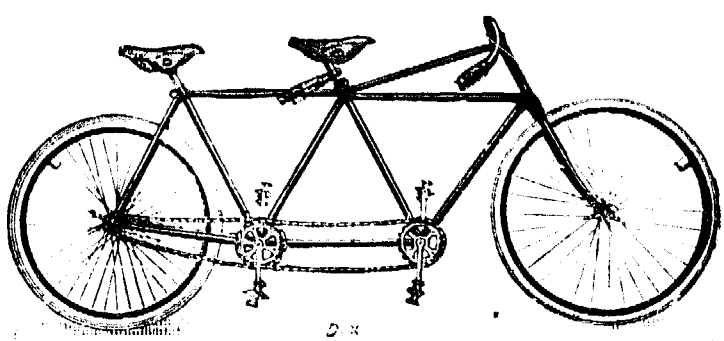


FIG. 140.

Figure 140 is an example of the present popular type of Tandem bicycle made by Messrs. Thomson and James. The machine is kinematically the same as that of figure 138, the particular difference being in the rear frame, which is of the diamond type, completely triangulated.